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REPRESENTATION THEORY OF HOPF GALOIS EXTENSIONS

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ABSTRACT

Let H be a Hopf algebra over the field k and $B \subset A$ a right faithfully flat right H-Galois extension. The aim of this paper is to study some questions of representation theory connected with the ring extension $B \subset A$, such as induction and restriction of simple or indecomposable modules. In particular, generalizations are given of classical results of Clifford, Green and Blattner on representations of groups and Lie algebras. The stabilizer of a left B-module is introduced as a subcoalgebra of H. Very often the stabilizer is a Hopf subalgebra. The special case when A is a finite dimensional cocommutative Hopf algebra over an algebraically closed field, B is a normal Hopf subalgebra and H is the quotient Hopf algebra was studied before by Voigt using the language of finite group schemes.

Introduction

Let H be a Hopf algebra over a field k and A a right *H-comodule algebra*, i.e. A is an algebra together with an H-comodule structure $\Delta_A : A \to A \otimes H$ such that Δ_A is an algebra map. Let B be the subalgebra of the H-coinvariant elements, $B := A^{\mathrm{co}\,H} := \{a \in A \mid \Delta_A(a) = a \otimes 1\}$. Assume that $B \subset A$ is an *H-Galois extension*. i.e. $A \otimes_B A \to A \otimes H$, $x \otimes y \mapsto x\Delta_A(y)$, is bijective, and that A is *faithfully flat* as right B-module.

A can be viewed as a non-commutative H-torsor over B (cf. [6]).

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The assumptions on the ring extension $B \subset A$ are strong enough to get non-trivial results and general enough to cover many interesting special cases.

Examples of faithfully flat H-Galois extensions $B \subset A$ include:

(1) G is a group, $G' \subset G$ a normal subgroup, G/G' the quotient group and $B := k[G'] \subset A := k[G]$, $H := k[G/G']$ are the group algebras.

(2) $\mathfrak{g}$ is a (p-)Lie algebra, $\mathfrak{g}' \subset \mathfrak{g}$ a (p-)Lie ideal, $\mathfrak{g}/\mathfrak{g}'$ the quotient (p-)Lie algebra and $B := U\mathfrak{g}' \subset A := U\mathfrak{g}$, $H := U\mathfrak{g}/\mathfrak{g}'$ are the universal (p-)enveloping algebras.

(3) G is a finite group scheme, G' a normal subgroup, G/G' the quotient group scheme and $B := H(G') \subset A := H(G)$, $H := H(G/G')$ are the covariant cocommutative Hopf algebras of the group schemes. If $\mathrm{Sp}(R)$ is a finite group scheme, i.e. R is a commutative finite dimensional Hopf algebra, then $H(\mathrm{Sp}(R))$ is defined as the dual Hopf algebra R^*. For representation theory of finite group schemes see the pioneering work of Voigt [28].

(4) A is any Hopf algebra, B a Hopf subalgebra which is normal (i.e. $B^+A = AB^+$) and such that A is right faithfully flat over B (this holds, for example, if A is pointed or cocommutative or if only its coradical is cocommutative [2]) and $H := A/AB^+$ is the quotient Hopf algebra. Of course, (1) and (2) are examples of (4), and (3) is the special case of (4) when A is finite dimensional and cocommutative.

(5) G is a group, $H := k[G]$ the group algebra, A any strongly G-graded algebra, $B := A_1$ (cf. [5]).

(6) B is any algebra, H any Hopf algebra with bijective antipode, H acts (weakly) on B, $\sigma : H \otimes H \to B$ is an invertible 2-cocycle, and $A := B \#_\sigma H$ is the *crossed product* (cf. [11], [3]). This example generalizes twisted group rings and skew group rings.

(7) $A = B \#_\sigma H$ as in (6), but replace B by $B \#_\sigma H'$ and H by H/HH'^+, where $H' \subset H$ is a normal Hopf subalgebra as in (4).

It is the philosophy of this paper that theorems become more general and proofs more natural if one studies faithfully flat Hopf Galois extensions in general instead of any one of the above examples. Very often, the methods of proof in the general case are very different from those of various particular cases. Sometimes, only a weakened version can be generalized, and the proof of an easy statement in a special case can be quite hard in general.

Most of the following depends on technical properties of Hopf Galois extensions proved in the first section.

In Section 2, a weakened version of the classical theorem of Clifford on restriction of simple representations is proved: If N is a simple A-module, then N need not be semisimple over B, but still its B-composition factors are of the form $A_g \otimes_B M$, g a group-like element of H (2.2). Here M is assumed to be a simple B-submodule of N, H is pointed and $A_g := \{a \in A \mid \Delta_A(a) = a \otimes g\}$. In the situation of example (3), 2.2 was proved by Voigt [28], 9.6, 9.7 and 9.8, in case H is irreducible, and in the more special case of p-Lie algebras by Diethelm-Nüssli in her thesis [7], Satz 3.6.

In Section 3, the notion of a stable B-module is introduced (cf. [4], p. 269, for group algebras). If M is B-finitely presented or H is finite dimensional, then the main Theorem 3.6 characterizes stable modules as follows: M is stable if and only if the A-endomorphism ring of the induced module $A \otimes_B M$ is an H-crossed product over the B-endomorphism ring of M. In the case of strongly G-graded algebras, 3.6 was proved by Dade [5], 5.24, generalizing previous results of Tucker, Conlon and Ward (cf. [4], §11C). In the situation of example (3), the implication of 3.6 where M is assumed to be stable was proved by Voigt using another description of stability and of crossed products (in the theory of group schemes). His proof [28], 12.6, is completely different from the general Hopf algebraic proof below.

In the next section, the existence of the stabilizer as a Hopf algebra is shown in the general case under rather weak assumptions (4.4). This was done by Voigt [28] for finite group schemes over algebraically closed fields (showing the representability of the corresponding functor), and by Blattner [2] for Lie algebras. In the group algebra case, the existence of the stabilizer is evident. In the general case, however, this is a non-trivial fact. The proof of 4.4 uses ideas of Greipel in his thesis [12], together with a simplification due to Takeuchi. It will be shown in another paper that 4.4 implies a general representability theorem if H is cocommutative.

In Section 5, induction of simple B-modules is studied. The main result 5.4 is a general version of Blattner's basic irreducibility criterion [2], Th. 3, for Lie algebras over algebraically closed fields in characteristic 0. Here, the Hopf algebra H is assumed to be cocommutative and pointed. 5.4 was proved by Voigt [28], 9.13, for finite group schemes over algebraically closed fields. Then Greipel [12] proved 5.4 for (not necessarily finite dimensional) cocommutative pointed Hopf algebras $B \subset A$, B normal. The proof of 5.4 in the general case is inspired by Greipel's procedure. As a corollary of 5.4, it is shown in 5.7 (as in [28]) that the computation of the Jordan–Hölder composition series of the induced module can be reduced to the stable case.

However, the role of the stabilizer in the determination of a Krull–Remak–Schmidt decomposition of the induced module seems to belong to the mysteries of the general (non-group algebra) case.

Finally, in the last section, some applications are given. The main result 6.2 is a version of Green's indecomposability theorem (cf. [4], (19.22)) for stable modules (see 6.4 for the case of simple modules). Here, the Hopf algebra H is assumed to be finite dimensional and local (or, more general, H^* is assumed to be pointed). Again, in the situation of example (3) when G/G' is unipotent, 6.2 was obtained by Voigt [28], 12.8, in a completely different way. Furthermore, some results on the block decomposition of A are derived in 6.5 and 6.6 generalizing statements of Morita [17] on the blocks of group algebras and of Voigt [28].

1. Conventions and technical results

Let k be a field and $\otimes = \otimes_k$. All algebras and coalgebras will be defined over k. Let H be a Hopf algebra, A a right H-comodule algebra and $B := A^{\mathrm{co}\,H}$. ${}_B\mathcal{M}$ will denote the category of left B-modules. If M is a left B-module, then the induced module $A \otimes_B M$ is a *Hopf module* in ${}_A\mathcal{M}^H$, where $a \otimes_B m \mapsto \Sigma\, a_0 \otimes_B m \otimes a_1$ is the H-comodule structure of the induced module. A Hopf module in ${}_A\mathcal{M}^H$ resp. $\mathcal{M}_A^H$ is a right H-comodule N which is also a left resp. right A-module such that the comodule structure map $\Delta_N : N \to N \otimes H$ is A-linear, i.e. (in Sweedler's notation) $\Delta_N(an) = \Sigma\, a_0 n_0 \otimes a_1 n_1$ resp. $\Delta_N(na) = \Sigma\, n_0 a_0 \otimes n_1 a_1$ for all $a \in A$ and $n \in N$ (cf. [22]). Morphisms of Hopf modules are A-linear and H-collinear maps.

1.1. Remark. (1) The canonical map can: $A \otimes_B A \to A \otimes H$, $\mathrm{can}(x \otimes_B y) = \Sigma\, x y_0 \otimes y_1$, is a morphism of Hopf modules in ${}_A\mathcal{M}^H$ and in $\mathcal{M}_A^H$, where

(a) $A \otimes_B A$ and $A \otimes H$ are Hopf modules in ${}_A\mathcal{M}^H$ with left A-module structures $a(x \otimes_B y) := ax \otimes_B y$, $a(x \otimes h) := ax \otimes h$ and right H-comodule structures $x \otimes_B y \mapsto \Sigma\, x_0 \otimes_B y \otimes x_1$, $x \otimes h \mapsto \Sigma\, x_0 \otimes h_2 \otimes x_1 S(h_1)$,

(b) $A \otimes_B A$ and $A \otimes H$ are Hopf modules in $\mathcal{M}_A^H$ with right A-module structures $(x \otimes_B y)a := x \otimes_B ya$, $(x \otimes h)a := \Sigma\, xa_0 \otimes ha_1$ and right H-comodule structures $x \otimes_B y \mapsto \Sigma\, x \otimes_B y_0 \otimes y_1$, $x \otimes h \mapsto \Sigma\, x \otimes h_1 \otimes h_2$ $(a, x, y \in A$ and $h \in H)$.

Proof. (a) If X is a right H-comodule and Y a left H-comodule, then $X \otimes Y$ is a right H-comodule by diagonal action in the folowing way: $x \otimes y \mapsto \Sigma\, x_0 \otimes y_0 \otimes x_1 S(y_{-1})$, where $\Delta_X(x) = \Sigma\, x_0 \otimes x_1$, $\Delta_Y(y) = \Sigma\, y_{-1} \otimes y_0$,

$x \in X$ and $y \in Y$. In particular, $A \otimes H$ is a right H-comodule as defined in (a). It is easily checked that $A \otimes_B A$ and $A \otimes H$ are Hopf modules in ${}_A\mathcal{M}^H$. The canonical map is trivially left A-linear. It is also right H-collinear, since

$$\Delta_{A\otimes H}(\Sigma\, xy_0 \otimes y_1) = \Sigma\, x_0 y_0 \otimes y_3 \otimes x_1 y_1 S(y_2) = \Sigma\, x_0 y_0 \otimes y_1 \otimes x_1.$$

(b) is easy to check (and well known).

(2) If the antipode of H is a bijective, and $B \subset A$ is any right H-Galois extension, then the following are equivalent:

(a) A is faithfully flat as right or left B-module.

(b) B is a direct summand in A as right or left B-module.

(c) A is injective as right H-comodule.

Proof. [22], Th. I; [10], 2.4.

(3) Let A be a right comodule algebra and assume that there is a right H-collinear and invertible (with respect to convolution) map $j: H \to A$. Define $B := A^{\operatorname{co} H}$. Then $B \subset A$ is an H-crossed product, $B \#_\sigma H \cong A$, $b \# h \mapsto bj(h)$. In particular, $B \subset A$ is a right H-Galois extension, and A is faithfully flat (free) as left B-module. If the antipode of H is bijective, then A is also right B-free (cf. [24], [3], [11]).

(4) Let A be a Hopf algebra with cocommutative coradical. Let B be a Hopf subalgebra which is normal in A, i.e. $B^+A = AB^+$ (B^+ is the kernel of the augmentation map of B). Then A is faithfully flat as left and right B-module. Let $H := A/AB^+$ be the quotient Hopf algebra and $A \to H$, $a \mapsto \bar{a}$, the canonical map. Then the diagonal of A induces a right H-comodule structure on A by $a \mapsto \Sigma\, a_1 \otimes \overline{a_2}$, and $B \subset A$ is an H-Galois extension.

Proof. By [26], Th. 3.1, and [27], Th. 1, A is faithfully flat over B on both sides, and $B = A^{\operatorname{co} H}$. Then can: $A \otimes_B A \to A \otimes H$ is bijective with inverse mapping $x \otimes \bar{y} \mapsto \Sigma\, xS(y_1) \otimes_B y_2$.

Throughout this paper, the following will be assumed:

H is a Hopf algebra and $B \subset A$ is a right H-Galois extension such that A is faithfully flat as right B-module.

Examples of this situation are given in the introduction (cf. 1.1).

Definition. Let V be a right H-comodule, $C \subset H$ a subcoalgebra and g a group-like element of H. Define $V(C) := \Delta_V^{-1}(V \otimes C)$ ($\cong V \square_H C$),

$$V_g := V(kg) = \{v \in V \mid \Delta_V(v) = v \otimes g\}.$$

If W is a left H-comodule, define ${}_gW := \{w \in W \mid \Delta_W(w) = g \otimes w\}$.

$V(C)$ is a right C-comodule and an H-subcomodule of V. Note that $A(C)$ is a left and right B-submodule of A. If $H' \subset H$ is a Hopf subalgebra, then $B \subset A(H')$ is a right H'-comodule extension and $A(H')_B$ is faithfully flat ([22], 3.11, 2)).

The following proposition is a fundamental observation for faithfully flat Hopf Galois extensions.

1.2. PROPOSITION. *Let $D \subset C$ be subcoalgebras of H, M a left B-module and V a left H-comodule. Then the canonical map*

$$\text{can}: (A(C)/A(D) \square_H V) \otimes_B M \to (A(C)/A(D) \otimes_B M) \square_H V,$$

$\text{can}(\Sigma\, \bar{a}_i \otimes v_i \otimes m_i) := \Sigma\, \bar{a}_i \otimes m_i v_i$, *is bijective. Here, $A(C)/A(D) \square_H V$ is a B-subbimodule of $A(C)/A(D) \otimes V$, where $b(\bar{a} \otimes v) := b\bar{a} \otimes v$, $(\bar{a} \otimes v)b := \overline{ab} \otimes v$ define the B-module structures, and $A(C)/A(D) \otimes_B M$ has right H-comodule structure $\bar{a} \otimes m \mapsto \Sigma\, \bar{a}_0 \otimes m \otimes a_1$.*

PROOF. Since A is right faithfully flat over B, it is enough to show that $A \otimes_B \text{can}$ is bijective. The canonical isomorphism $A \otimes_B A \to A \otimes H$ induces an isomorphism $\text{can}(C): A \otimes_B A(C) \to A \otimes C$ by cotensoring with $- \square_H C$. Note that $(A \otimes_B A) \square_H C \cong A \otimes_B A(C)$, since A is B-flat. The same isomorphism exists for D. Hence by the flatness of A over B, $\Phi: A \otimes_B A(C)/A(D) \to A \otimes C/D$, $\Phi(x \otimes \bar{y}) := \Sigma\, xy_0 \otimes \overline{y_1}$, is bijective. Now it is easily seen that $A \otimes_B \text{can}$ is the composition of the following canonical isomorphisms:

$$\begin{aligned}
&A \otimes_B (A(C)/A(D) \square_H V) \otimes_B M \\
&\quad \cong ((A \otimes_B A(C)/A(D)) \square_H V) \otimes_B M \quad && (\text{since } A_B \text{ is flat}) \\
&\quad \cong ((A \otimes C/D) \square_H V) \otimes_B M && (\text{induced by } \Phi) \\
&\quad \cong A \otimes_B M \otimes (C/D \square_H V) \\
&\quad \cong ((A \otimes C/D) \otimes_B M) \square_H V && (\text{induced by } \Phi) \\
&\quad \cong (A \otimes_B A(C)/A(D) \otimes_B M) \square_H V \\
&\quad \cong A \otimes_B ((A(C)/A(D) \otimes_B M) \square_H V) && (\text{since } A_B \text{ is flat}).
\end{aligned}$$

1.3. Corollary. *Assume the situation of* 1.2.

(1) $A(C)/A(D)\,\square_H\, V$ *is faithfully flat as right B-module.*

(2) $A\otimes_B M$ *is faithfully coflat as right H-comodule.*

(3) *The canonical map* $A(C)\otimes_B M \to (A\otimes_B M)\,\square_H\, C$ *is a B-linear isomorphism, and* $A(C)\otimes_B M$ *is injective* (= *coflat*) *as right C-comodule.*

(4) *The canonical map* $A(C)\otimes_B M \to A\otimes_B M$, *defined by the inclusion* $A(C)\subset A$, *is injective.*

(5) *The left B-linear map* $M\to (A\otimes_B M)^{\mathrm{co}\,H}$, $m\mapsto 1\otimes_B m$, *is bijective.*

Proof. (1) By the proof of 1.2,

$$M\mapsto A\otimes_B (A(C)/A(D)\,\square_H\, V)\otimes_B M \cong A\otimes_B M\otimes (C/D\,\square_H\, V)$$

preserves and reflects exactness, since A_B is faithfully flat. Hence, $A(C)/A(D)\,\square_H\, V$ is faithfully flat over B, since A_B is faithfully flat.

(2) is proved in the same way as (1), since $A\otimes_B((A\otimes_B M)\,\square_H\, V)\cong A\otimes_B M\otimes V$ (take $C := H$ and $D := 0$ in the proof of 1.2).

(3) follows from 1.2 and (2).

(4) follows from (3).

(5) Let k be the trivial H-comodule. Then, by 1.2, $(A\otimes_B M)^{\mathrm{co}\,H}\cong (A\otimes_B M)\,\square_H\, k\cong (A\,\square_H\, k)\otimes_B M$. This proves (5), since $A\,\square_H\, k\cong B$.

The next corollary permits the calculation of the B-module structure of restricted modules in certain cases. It will be applied in Sections 2 and 5.

1.4. Corollary. *Assume the situation of* 1.2. *Let g be a group-like element of H.*

(1) $A(C)/A(D)\otimes_B M\to (A\otimes_B M\otimes C/D)^{\mathrm{co}\,H}$, $\bar a\otimes_B m\mapsto \Sigma\, a_0\otimes_B m\otimes \overline{a_1}$, *is an isomorphism of left B-modules, where* $A\otimes_B M\otimes C/D$ *is a right H-comodule in the following way* (cf. 1.1): $a\otimes_B m\otimes \bar c\mapsto \Sigma\, a_0\otimes_B m\otimes \overline{c_2}\otimes a_1 S(c_1)$.

(2) $(A\otimes_B M\otimes_g (C/D))^{\mathrm{co}\,H}\cong A_g\otimes_B M\otimes_g (C/D)$ *as left B-modules. Here,* $A\otimes_B M\ \otimes_g (C/D)$ *is viewed as a right H-subcomodule of the comodule* $A\otimes_B M\otimes C/D$ *in* (1).

(3) *If C is cocommutative, then* $A(C)/A(D)_g\otimes_B M\to A_g\otimes_B M\otimes (C/D)_g$, $\bar a\otimes_B m\mapsto \Sigma\, a_0\otimes_B m\otimes\overline{a_1}$, *is an isomorphism of left B-modules.*

Proof. (1) The canonical map $A\otimes_B A(C)/A(D)\otimes_B M\to A\otimes_B M\otimes C/D$, $x\otimes_B \bar y\otimes_B m\mapsto \Sigma\, xy_0\otimes_B m\otimes\overline{y_1}$, is bijective (see the proof of 1.2). This map is right H-collinear, where the module on the left has the usual H-comodule structure defined by Δ_A and the comodule structure on the right is described in

1.4. Hence, the above map induces an isomorphism of the H-coinvariant elements. By 1.3 (5), the coinvariant elements on the left are $A(C)/A(D)\otimes_B M$.

(2) Let $\Sigma\, a_i \otimes_B m_i \otimes \bar{c}_i$ be an element of $A\otimes_B M\otimes_g (C/D)$. This element is H-coinvariant if and only if $\Sigma\, a_{i0}\otimes_B m_i \otimes \bar{c}_i \otimes a_{i1} g^{-1} = \Sigma\, a_i \otimes_B m_i \otimes \bar{c}_i \otimes 1$, or equivalently, $\Sigma\, a_{i0}\otimes_B m_i\otimes\bar{c}_i\otimes a_{i1} = \Sigma\, a_i\otimes_B m_i\otimes\bar{c}_i\otimes g$. This proves the claim, since $(A\otimes_B M)_g \cong A_g\otimes_B M$, by 1.3(3) (take $C := kg$).

(3) By the proof of 1.2 (take $V := kg$), the canonical map

$$A\otimes_B (A(C)/A(D))_g \otimes_B M \to A\otimes_B M\otimes (C/D)_g$$

is bijective. Then $A(C)/A(D)_g\otimes_B M$ is B-isomorphic to $(A\otimes_B M\otimes (C/D)_g)^{\mathrm{co}\,H}$ by the same argument as before. But $(C/D)_g = {}_g(C/D)$, since C is cocommutative. Hence, (2) can be applied, and the isomorphism in (3) follows.

1.5. Corollary. *Assume the situation of* 1.2. *Let g, h be group-like elements and C_i, $i\in I$, subcoalgebras of H.*

(1) *The multiplication maps $A\otimes_B A_h\to A$ and $A_g\otimes_B A_h\to A_{gh}$ are isomorphisms.*

(2) ${}_B\mathcal{M}\to {}_B\mathcal{M}$, $M\mapsto A_g\otimes_B M$, *is an equivalence of categories.*

(3) $A_gA(C) = A(gC)$.

(4) $\Sigma_i A(C_i) = A(\Sigma_i C_i)$.

Proof. (1) As in the proof of 1.2 (take $C := kh$), the canonical map $A\otimes_B A_h\to A\otimes kh$, $x\otimes_B y\mapsto xy\otimes h$, is bijective. Hence, the multiplication map $A\otimes_B A_h\to A$ is bijective for all group-like elements h. Therefore, multiplication defines an isomorphism $A\otimes_B A_g\otimes_B A_h\to A\otimes_B A_{gh}$ (apply the above to g, h and gh). Hence, $A_g\otimes_B A_h\to A_{gh}$ is bijective, since A_B is faithfully flat.

(2) By (1), the multiplication maps $A_g\otimes_B A_{g^{-1}}\to B$, $A_{g^{-1}}\otimes_B A_g\to B$ are isomorphisms, and A_g is an invertible B-bimodule.

(3) Clearly, $A_gA(C)\subset A(gC)$ for all group-like elements g. Therefore, $A_gA_{g^{-1}}A(gC)\subset A_gA(C)$. But by (1), $A_gA_{g^{-1}} = B$. Hence, $A_gA(C) = A(gC)$.

(4) Consider the canonical surjective map $\oplus_i C_i\to\Sigma_i C_i$. This map is left H-collinear. By 1.3(2), A is coflat as right H-comodule. Hence $A\,\square_H(\oplus_i C_i)\to A\,\square_H(\Sigma_i C_i)$ is surjective. This proves the claim, since the cotensor product commutes with direct sums.

In the proof of the general version of Blattner's theorem in Section 5, the following result on induction of modules over $A(H')$, H' a Hopf subalgebra, will be needed.

1.6. Proposition. *Let $H' \subset H$ be a Hopf subalgebra. Assume that the antipode of H' is bijective and that H is faithfully flat as left H'-module. Let $C \subset H$ be a subcoalgebra such that $CH' \subset C$. Define $A' := A(H')$ and $\overline{H} := H/HH'^{+}$. Let Q be a left A'-module. Then $A(C)$ is a (B, A')-bimodule, $A \otimes_{A'} Q$ is a right $\overline{H}$-comodule with structure map $a \otimes_{A'} q \mapsto \Sigma\, a_0 \otimes_{A'} q \otimes \overline{a_1}$, and the left B-linear map $A(C) \otimes_B Q \to (A(C) \otimes_{A'} Q) \,\square_{\overline{H}}\, H$, $a \otimes_B q \mapsto \Sigma\, a_0 \otimes_{A'} q \otimes a_1$, is bijective.*

Proof. By [27], Th. 1, $\mathcal{M}^H_{H'} \to \mathcal{M}^{\overline{H}}$, $X \mapsto X \otimes_{H'} k$, is an equivalence with quasi-inverse $Y \mapsto Y \square_{\overline{H}} H$, since H is left faithfully flat over H'. In particular, H is faithfully coflat as left $\overline{H}$-comodule, and $H \otimes H' \to H \square_{\overline{H}} H$, $x \otimes y \mapsto \Sigma\, x_1 \otimes x_2 y$, is bijective, since $H \otimes H' \otimes_{H'} k \cong H \cong (H \square_{\overline{H}} H) \otimes_{H'} k$. Hence, by cotensoring with $C \square_{H}-$, $C \otimes H' \to C \square_{\overline{H}} H$, $x \otimes y \mapsto \Sigma\, x_1 \otimes x_2 y$, is bijective. Since the antipode of H' is bijective, $C \otimes H' \to C \otimes H'$, $x \otimes y \mapsto \Sigma\, xy_1 \otimes S(y_2)$, is an isomorphism. The composition of both maps gives the isomorphism $\Psi : C \otimes H' \to C \square_{\overline{H}} H$, $\Psi(x \otimes y) := \Sigma\, x_1 y \otimes x_2$.

As in the proof of 1.2, let $\mathrm{can}(D) : A \otimes_B A(D) \to A \otimes D$ be the canonical isomorphism. Finally, let Φ be the canonical map in 1.6. It suffices to show the bijectivity of Φ in case $Q = A'$, since both sides of Φ are right exact functors in Q (H is $\overline{H}$-coflat) which commute with arbitrary direct sums.

As before, it will suffice to write $A \otimes_B \Phi$ as a composition of isomorphisms:

$$\begin{aligned}
A \otimes_B A(C) \otimes_B A' &\cong (A \otimes C) \otimes_B A' && \text{(induced by can}(C))\\
&\cong A \otimes_B A' \otimes C && \text{(permute } C \text{ and } A')\\
&\cong A \otimes H' \otimes C && \text{(induced by can}(H'))\\
&\cong A \otimes C \otimes H' && \text{(permute } H' \text{ and } C)\\
&\cong A \otimes C \square_{\overline{H}} H && \text{(induced by } \Psi)\\
&\cong (A \otimes_B A(C)) \square_{\overline{H}} H && \text{(induced by can}(C))\\
&\cong A \otimes_B (A(C) \square_{\overline{H}} H) && \text{(associativity holds, since } A_B \text{ is flat).}
\end{aligned}$$

1.7. Corollary. *Assume the situation in* 1.6.

(1) *The inclusion $A(C) \subset A$ induces an injective map $A(C) \otimes_{A'} Q \to A \otimes_{A'} Q$.*

(2) *If the antipode of H is bijective, then* $\mathrm{can}' : A \otimes_{A'} A \to A \otimes \overline{H}$, $\mathrm{can}'(x \otimes_{A'} y) := \Sigma\, x_0 y \otimes \overline{x_1}$, *is bijective.*

(3) *$A(C)$ is faithfully flat as right A'-module.*

Proof. (1) follows from 1.6, since $A(C) \otimes_B Q \to A \otimes_B Q$ is injective by 1.3(4), and since H is faithfully coflat as left $\overline{H}$-comodule.

(2) The canonical map can$'$: $A \otimes_B A \to A \otimes H$ is bijective, since the antipode of H is bijective. Now (2) follows from 1.6 in the same way as (1).

(3) follows from 1.6, since A_B is faithfully flat and H is faithfully coflat over $\overline{H}$.

1.8. REMARK. (1) Assume in 1.6 that the antipode of H is bijective. Then $A_{A'}$ is faithfully flat and can$'$: $A \otimes_{A'} A \to A \otimes \overline{H}$ is bijective by 1.7. Therefore, cf. [22], 3.7, $A' = A^{\mathrm{co}\,\overline{H}}$, and the induction functor ${}_{A'}\mathcal{M} \to {}_{A}\mathcal{M}^{\overline{H}}$, $Q \mapsto A \otimes_{A'} Q$, is an equivalence with quasi-inverse $N \mapsto N^{\mathrm{co}\,\overline{H}}$. In particular, $Q \cong (A \otimes_{A'} Q)^{\mathrm{co}\,\overline{H}}$.

(2) Assume in 1.6 that H is finite dimensional. Then $H \cong \overline{H} \otimes H'$ as left comodules over $\overline{H}$ (and right H'-comodules). This follows from the theorem of Nichols and Zoeller [18], see [23]. Thus $A(C) \otimes_B Q \cong (A(C) \otimes_{A'} Q)^n$ as left B-modules, where n is the dimension of H'.

2. Restriction of induced and of simple modules

Let $(H_n)_{n \geqq 0}$ be the coradical filtration of H ([24], §9.1 and §11.1): H_0 is the coradical, H is the union of the H_n, and $\Delta(H_n) \subset \Sigma_{i=0}^{n} H_i \otimes H_{n-i}$. Hence, the comultiplication of H induces a left and right H_0-comodule structure on H_{n+1}/H_n, $n \geqq 0$. If V is any right H-comodule, then $V_n := V(H_n) \cong V \square_H H_n$, $n \geqq 0$, defines the *coradical filtration on* V.

2.1. THEOREM. *Assume H is pointed. Let M be a left B-module and $N := A \otimes_B M$ the induced module. Let $i_n : A_n \to A$, $n \geqq 0$, be the inclusion maps, where (A_n) is the coradical filtration. Then $i_n \otimes_B M : A_n \otimes_B M \to A \otimes_B M$ is injective, and $N_n :=$ image of $i_n \otimes_B M$, $n \geqq 0$, defines a filtration of N as B-module (by restriction) such that $A \otimes_B M = \bigcup_n N_n$, and each quotient N_{n+1}/N_n is a B-direct sum of modules isomorphic to $A_g \otimes_B M$, g a group-like element of H.*

PROOF. By 1.3(4), $i_n \otimes_B M$ is injective. The equality $N = \bigcup N_n$ follows from $A = \bigcup A_n$. Take $D := H_n \subset C := H_{n+1}$ in 1.4. By 1.4(1), N_{n+1}/N_n is B-isomorphic to $(A \otimes_B M \otimes H_{n+1}/H_n)^{\mathrm{co}\,H}$. Let G be the set of group-like elements of H. Then $H_{n+1}/H_n = \oplus_g (H_{n+1}/H_n)$ is a decomposition of left H-comodules. Hence,

$$N_{n+1}/N_n \cong \bigoplus_g (A \otimes_B M \otimes_g (H_{n+1}/H_n)^{\mathrm{co}\,H}$$

$$\cong \bigoplus_g A_g \otimes_B M \otimes_g (H_{n+1}/H_n) \qquad \text{(by 1.4(2))}$$

is B-isomorphic to a direct sum of copies of $A_g \otimes_B M$, $g \in G$.

2.2. Corollary. *Assume H is pointed. Let N be a simple left A-module and M a simple B-submodule of N. Then there is a sequence*

$$N_0 \subset N_1 \subset N_2 \cdots \text{ of } B\text{-submodules of } N,\ N = \bigcup N_n,$$

such that each quotient N_{n+1}/N_n, $n \geqq 0$, is a B-direct sum of simple B-modules isomorphic to $A_g \otimes_B M$, g a group-like element of H.

In particular, if N is a B-module of finite length, then each B-composition factor of N is of the form $A_g \otimes_B M$, g a group-like element of H.

Proof. The multiplication map $A \otimes_B M \to AM = N$ is an epimorphism of $B -$ (and $A -$) modules. Since any $A_g \otimes_B M$ is B-simple by 1.5(2), everything follows from 2.1.

2.3. Remark. Let G be a finite group and $G' \subset G$ a normal subgroup. Let N be a simple left module over the group algebra $A := k[G]$ and M a simple $k[G']$-submodule of N, $B := k[G']$. Let $\bar{g}$ be the image of $g \in G$ in G/G'. If X is any left $k[G']$-module, let ${}_gX$ be X as k-module with the twisted G'-operation $(g', x) \mapsto g^{-1}g'g \cdot x$. Note that the coset gG' is a basis of $A_{\bar{g}}$. Thus $A_{\bar{g}} \cong {}_gk[G']$, and $A_{\bar{g}} \otimes_g M \cong {}_gM$ as left $k[G']$-modules. Hence, by 2.2, all G'-composition factors of N are of the form ${}_gM$, $g \in G$. Therefore, 2.2 can be viewed as a generalization of a weakened version of Clifford's theorem which says furthermore that N is semisimple as $k[G']$-module.

3. Endomorphism ring of induced modules and stable modules

In this section, k will be a commutative ring. As always, $B \subset A$ is a right H-Galois extension and A is faithfully flat as right B-module.

Assume furthermore the following *situation* (S):

Let M be a left B-module. Assume M is B-finitely presented and H is flat over k or H is finitely generated and projective as k-module.

Let $E' := \mathrm{End}_B(M)^{\mathrm{op}}$ and $E := \mathrm{End}_A(A \otimes_B M)^{\mathrm{op}}$ be the dual endomorphism rings. The induced module $N := A \otimes_B M$ is a Hopf module in ${}_A\mathcal{M}^H$ as described in Section 1. N is also a right E-module by $nF := F(n)$, $n \in N$, $F \in E$.

3.1. Lemma. *Assume situation (S). Then there is exactly one right H-comodule algebra structure Δ_E on E such that N is a Hopf module in $\mathcal{M}_E^H$, i.e.*

$\Delta_N(nF) = \Sigma\, n_0 F_0 \otimes n_1 F_1$ for all $n \in N$ and $F \in E$. The algebra homomorphism $E' \to E^{\mathrm{co}\,H}$, $f \mapsto \mathrm{id} \otimes f$, is bijective.

PROOF. By assumption (S), the canonical map

$$\mathrm{can} : \mathrm{Hom}_B(M, N) \otimes H \to \mathrm{Hom}_B(M, N \otimes H) \text{ is bijective.}$$

Define Δ_E as the composition

$$E \cong \mathrm{Hom}_B(M, N) \xrightarrow{\mathrm{Hom}_B(M,\Delta_N)} \mathrm{Hom}_B(M, N \otimes H) \xleftarrow{\mathrm{can}} \mathrm{Hom}_B(M, N) \otimes H$$
$$\cong E \otimes H.$$

By definition, for all $F \in E$ and $m \in M$: $\Sigma\, F_0(1 \otimes m) \otimes F_1 = \Delta_N(F(1 \otimes m))$, where $\Delta_E(F) = \Sigma\, F_0 \otimes F_1$. Hence, for all $a \in A$:

$$\Delta_N(F(a \otimes m)) = \Delta_N(aF(1 \otimes m)) = \Sigma\, F_0(a_0 \otimes m) \otimes a_1 F_1,$$

that is, N is a Hopf module. Uniqueness of Δ_E follows from the first equation, and it is easily checked that (E, Δ_E) is in fact an H-comodule algebra.

The mapping from E' to E is injective, since A_B is faithfully flat. Let F be in E. Then $F \in E^{\mathrm{co}\,H}$ if and only if, for all $m \in M$, $\Delta_N(F(1 \otimes m)) = F(1 \otimes m) \otimes 1$. By 1.3(5), $M \cong (A \otimes_B M)^{\mathrm{co}\,H}$. Hence, $E' \to E^{\mathrm{co}\,H}$ is bijective.

In this section, properties of the induced module will be characterized by properties of the H-comodule algebra E.

DEFINITION. Let M be a left B-module. M will be called *A-stable* or *stable* if there is a left B-linear and right H-collinear isomorphism

$$A \otimes_B M \cong M \otimes H.$$

Here,

$$b(a \otimes_B m) := ba \otimes_B m \quad \text{and } a \otimes_B m \mapsto \Sigma\, a_0 \otimes_B m \otimes a_1, \quad \text{resp.}$$
$$b(m \otimes h) := bm \otimes h \qquad \text{and } m \otimes h \mapsto m \otimes \Delta(h)$$

are the B-module and H-comodule structures of $A \otimes_B M$ (as above) resp. of $M \otimes H$.

3.2. REMARK. (1) Let G be a group and $G' \subset G$ a normal subgroup. Consider the $k[G/G']$-Galois extension $B := k[G'] \subset A := k[G]$, $\Delta_A(g) = g \otimes \bar{g}$ for all $g \in G$. In this case, M is stable if and only if, for all $g \in G$, M is B-isomorphic to the twisted B-module ${}_gM$. Hence, the above definition generalizes the well-known notion of stable modules in representation theory of groups.

(2) In the case of finite group schemes (over an algebraically closed field) in example (3) of the introduction, M is stable if and only if M is stable in the sense of Voigt [28]. This follows from [28], 7.4.

(3) Assume the antipode of H is bijective. Then any left A-module N is stable over B (by restriction).

PROOF. Since the antipode is bijective, $\mathrm{can}': A \otimes_B A \to A \otimes H$, $x \otimes y \mapsto \Sigma x_0 y \otimes x_1$, is an isomorphism. Hence, tensoring with $- \otimes_A N$ gives an isomorphism $A \otimes_B N \to N \otimes H$ of B-modules and H-comodules.

(4) Assume in the above definition that M, B and H are finite dimensional over the field k. Then the following are equivalent (cf. [28], 7.4):

(a) M is stable.

(b) There is a B-linear isomorphism $A \otimes_B M \cong M^t$ for some natural number t.

PROOF. (a)$\Rightarrow$(b) is trivial. (b)$\Rightarrow$(a): Let $N := A \otimes_B M$. By (3), there is a B-linear and H-collinear isomorphism $A \otimes_B N \cong N \otimes H$. By (b), $N \cong M^t$ over B. Hence, $(A \otimes_B M)^t \cong (M \otimes H)^t$ as left B-modules and right H-comodules, i.e. as left $B \otimes H^*$-modules. By Krull–Remak–Schmidt, $A \otimes_B M \cong M \otimes H$ as B-modules and H-comodules.

(5) If M is stable, there is an isomorphism $\Phi: A \otimes_B M \to M \otimes H$ of left B-modules and right H-comodules such that $\Phi(1 \otimes m) = m \otimes 1$ for all $m \in M$ (i.e. Φ is *M-compatible*).

PROOF. Let $\Phi: A \otimes_B M \to M \otimes H$ be a B-linear and H-collinear isomorphism. Since, by 1.3(5), $M \cong (A \otimes_B M)^{\mathrm{co}\,H}$, the H-collinear map Φ induces a B-linear isomorphism $f: M \to M$. Then $(f^{-1} \otimes \mathrm{id})\Phi$ is B-linear, H-collinear and M-compatible.

3.3. THEOREM. *Assume situation (S). If the antipode of H is bijective, then the following are equivalent:*

(1) *$M \to A \otimes_B M$, $m \mapsto 1 \otimes m$, is a B-split monomorphism.*

(2) *M is B-isomorphic to a B-direct summand of an A-module.*

(3) *M is B-isomorphic to a direct summand of a stable module.*

(4) *There is a right H-collinear and unitary map $J: H \to E$.*

PROOF. (1)$\Rightarrow$(2) is trivial, and (2)$\Rightarrow$(3) follows from 3.2(3).

(3)$\Rightarrow$(4): Let V be a stable left B-module and U a left B-module such that

$M \oplus U \cong V$ over B. Then $E \cong \mathrm{Hom}_B(M, A \otimes_B M)$ is an H-comodule direct summand of $\mathrm{Hom}_B(M, A \otimes_B V)$, where the comodule structures are defined as in 3.1 via the comodule structures of $A \otimes_B M$ and $A \otimes_B V$. Since V is stable, $\mathrm{Hom}_B(M, A \otimes_B V)$ is isomorphic to $\mathrm{Hom}_B(M, V) \otimes H$ as comodules, where $\mathrm{id} \otimes \Delta$ is the comodule structure on the right. Hence E as an H-comodule is injective with respect to k-split monomorphisms of comodules. In particular, there is a right H-collinear map $\varphi : E \otimes H \to E$ such that $\varphi \Delta_E = \mathrm{id}$. Hence, $J : H \to E$, $J(h) := \varphi(1 \otimes h)$, is right H-collinear and unitary.

(4)$\Rightarrow$(1) (here, the antipode of H need not be bijective): Define $q : A \otimes_B M \to M$ by the equation $1 \otimes q(n) = \Sigma\, n_0 J(S(n_1))$ in $A \otimes_B M = N$ for all $n \in N$. By 3.1,

$$\Delta_N(\Sigma\, n_0 J(S(n_1))) = \Sigma\, n_0 J(S(n_3)) \otimes n_1 S(n_2) = \Sigma\, n_0 J(S(n_1)) \otimes 1.$$

Hence q is well-defined, since, by 1.3(5), $M \cong (A \otimes_B M)^{\mathrm{co}\, H}$. Clearly, q is left B-linear, and $q(1 \otimes m) = m$ for all $m \in M$, since J is unitary.

Note that condition (1) in 3.3 is clearly satisfied in the special case when B is a direct summand of A as a *B-bimodule*.

The following notation will be used in the sequel. For any $h \in H$ choose finitely many elements $r_i(h)$, $l_i(h) \in A$, $i \in I$, such that $1 \otimes h = \Sigma\, r_i(h) l_i(h)_0 \otimes l_i(h)_1$ in $A \otimes H$. Note that $\Sigma\, r_i(h) \otimes_B l_i(h)$ is uniquely determined as the inverse image of $1 \otimes h$ by the isomorphism can: $A \otimes_B A \to A \otimes H$.

3.4. Remark. (1) Take $A = H$ as comodule algebra via Δ. Then $B = k$ and $\Sigma\, r_i(h) \otimes l_i(h) = \Sigma\, S(h_1) \otimes h_2$. Hence, in general, $\Sigma\, r_i(h) \otimes_B l_i(h)$ should be viewed as a substitute of $\Sigma\, S(h_1) \otimes h_2$ or $g^{-1} \otimes g$, g a group-like element.

(2) For all $h, h' \in H$, $b \in B$ and $a \in A$ the following identities hold:

(a) $\Sigma\, b r_i(h) \otimes_B l_i(h) = \Sigma\, r_i(h) \otimes_B l_i(h) b$,

(b) $\Sigma\, a_0 r_i(a_1) \otimes_B l_i(a_1) = 1 \otimes_B a$,

(c) $\Sigma\, r_i(h) l_i(h) = \varepsilon(h)$,

(d) $\Sigma\, r_i(h) \otimes_B l_i(h)_0 \otimes l_i(h)_1 = \Sigma\, r_i(h_1) \otimes_B l_i(h_1) \otimes h_2$,

(e) $\Sigma\, r_i(h)_0 \otimes_B l_i(h) \otimes r_i(h)_1 = \Sigma\, r_i(h_2) \otimes_B l_i(h_2) \otimes S(h_1)$,

(f) $\Sigma\, r_i(hh') \otimes_B l_i(hh') = \Sigma\, r_i(h') r_j(h) \otimes_B l_j(h) l_i(h')$,

(g) $\Sigma\, r_i(h_1) \otimes_B l_i(h_1) r_j(h_2) \otimes_B l_j(h_2) = \Sigma\, r_i(h) \otimes_B 1 \otimes_B l_i(h)$.

Proof. (a) and (b) follow from the definition by applying can. The multiplication map factorizes as $(\mathrm{id} \otimes \varepsilon)$ can. This proves (c). To prove (d), apply can $\otimes$ id and use the collinearity of can as described in 1.1(b). Similarly,

(e) follows from the collinearity of can in the sense of 1.1(a). Finally, to prove (f) resp. (g), apply can resp. (can $\otimes$ id) (id $\otimes$ can) and use (d).

Let $\alpha : H \to k$ be an algebra homomorphism and N a left A-module. Then $\alpha_A : A \to A$, $\alpha_A(a) := \Sigma\, a_0\alpha(a_1)$, is an algebra isomorphism (with inverse $\alpha_A^{-1}(a) = \Sigma\, a_0\alpha(S(a_1))$), and ${}_\alpha N$ will denote the A-module induced by α_A, i.e. ${}_\alpha N = N$ as k-module with A-module structure $(a, n) \mapsto \alpha_A(a)n$.

3.5. Corollary. *Let X and Y be left A-modules. For all $\varphi \in \mathrm{Hom}_B(X, Y)$ and $h \in H$ define $\varphi \circ h : X \to Y$* by $(\varphi \circ h)(x) := \Sigma\, r_i(h)\varphi(l_i(h)x)$, $x \in X$. *Then*

(1) $\mathrm{Hom}_B(X, Y)$ *is a right H-module.*

(2) $\mathrm{Hom}_B(X, X)$ *is a right H-module algebra.*

(3) *For any algebra homomorphism $\alpha : H \to k$ and $\varphi \in \mathrm{Hom}_B(X, Y)$ the following are equivalent:*

(a) *For all $h \in H$: $\varphi \circ h = \varphi\alpha(h)$.*

(b) *$\varphi : X \to {}_\alpha Y$ is A-linear.*

(4) $\mathrm{Hom}_A(X, Y) = \mathrm{Hom}_B(X, Y)^H$ $(:= \{\varphi \in \mathrm{Hom}_B(X, Y) \mid$ *for all* $h \in H$: $\varphi \circ h = \varphi\varepsilon(h)\})$.

Proof. (1) Let $\varphi \in \mathrm{Hom}_B(X, Y)$. Since φ is B-linear, $(\varphi \circ h)(x)$ is well-defined. By 3.4(2)(a), $\varphi \circ h$ is B-linear. By 3.4(2)(f) and (c), $\varphi \circ (hh') = (\varphi \circ h) \circ h'$ and $\varphi \circ 1 = \varphi$.

(2) By 3.4(2)(g) and (c), $(\varphi\psi) \circ h = \Sigma(\varphi \circ h_1)(\psi \circ h_2)$ and $\mathrm{id} \circ h = \mathrm{id}\, \varepsilon(h)$.

(3) (a)$\Rightarrow$(b): By 3.4(2)(b), for all $a \in A$ and $x \in X$:

$$\begin{aligned}\varphi(ax) &= \Sigma\, a_0 r_i(a_1)\varphi(l_i(a_1)x) && \text{(by 3.4(2)(b))}\\ &= \Sigma\, a_0(\varphi \circ a_1)(x) \\ &= \Sigma\, a_0\alpha(a_1)\varphi(x) && \text{(by (a))}\\ &= \alpha_A(a)\varphi(x).\end{aligned}$$

(b)$\Rightarrow$(a): For all $h \in H$ and $x \in X$:

$$\begin{aligned}(\varphi \circ h)(x) &= \Sigma\, r_i(h)\varphi(l_i(h)x) \\ &= \Sigma\, r_i(h)l_i(h)_0\alpha(l_i(h)_1)\varphi(x) && \text{(by (b))},\\ &= \Sigma\, \varepsilon(h_1)\alpha(h_2)\varphi(x) && \text{(by 3.4(2)(d) and (c))}\\ &= \alpha(h)\varphi(x).\end{aligned}$$

(4) follows from (3).

REMARK. Take $A = H$ in 3.5. Then $(\varphi \circ g)(x) = g^{-1}\varphi(gx)$ for all group-like elements $g \in H$. Hence, 3.5 specializes to the well-known operation of group elements on homomorphisms by diagonal action in representation theory of groups. Of course, 3.5 holds for arbitrary (not necessarily faithfully flat) Hopf Galois extensions.

3.6. THEOREM. *Assume situation (S). Then the following are equivalent:*

(1) *M is stable.*

(2) *$E' \subset E$ is an H-crossed product (i.e. there is a right H-collinear and invertible map $J : H \to E$).*

PROOF. (1)$\Rightarrow$(2); By assumption, there is a left B-linear and right H-collinear isomorphism $\Phi : A \otimes_B M \to M \otimes H$. Let $\Psi := \Phi^{-1}$. Define $q : A \otimes_B M \to M$ by $q = (\mathrm{id} \otimes \varepsilon)\Phi$. Then q is B-linear, and $\Phi(a \otimes m) = \Sigma\, q(a_0 \otimes m) \otimes a_1$. Define $J, J' : H \to E$ by

$$J(h)(1 \otimes m) := \Psi(m \otimes h),$$

$$J'(h)(1 \otimes m) := \Sigma\, r_i(h) \otimes q(l_i(h) \otimes m).$$

Then J is H-collinear, since Ψ is H-collinear, J' is well-defined, since the right-hand side is B-linear in m by 3.4(2)(a). It remains to prove that J' is $*$-inverse to J.

For all $h \in H$ and $m \in M$:

$$\begin{aligned}(J' * J)(h)(1 \otimes m) &= \Sigma\, J(h_2)(J'(h_1)(1 \otimes m))\\ &= \Sigma\, r_i(h_1) J(h_2)(1 \otimes q(l_i(h_1) \otimes m))\\ &= \Sigma\, r_i(h_1)\Psi(q(l_i(h_1) \otimes m) \otimes h_2)\\ &= \Sigma\, r_i(h)\Psi(q(l_i(h)_0 \otimes m) \otimes l_i(h)_1) \qquad \text{(by 3.4(2)(d))}\\ &= \Sigma\, r_i(h) l_i(h) \otimes m\\ &= \varepsilon(h) 1 \otimes m \qquad \text{(by 3.4(2)(c))}.\end{aligned}$$

To prove $J' * J = \eta\varepsilon$, write $\Psi(m \otimes h) = \Sigma\, a_j \otimes m_j$ in $A \otimes_B M$. Then

$$\begin{aligned}(J*J')(h)(1\otimes m) &= \Sigma\, J'(h_2)(J(h_1)(1\otimes m))\\ &= \Sigma\, J'(h_2)(\Psi(m\otimes h_1))\\ &= \Sigma\, J'(a_{j1})(a_{j0}\otimes m_j) && \text{(since } \Psi \text{ is collinear)}\\ &= \Sigma\, a_{j0}r_i(a_{j1})\otimes q(l_i(a_{j1})\otimes m_j)\\ &= \Sigma\, 1\otimes q(a_j\otimes m_j) && \text{(by 3.4(2)(b))}\\ &= 1\otimes q(\Psi(m\otimes h))\\ &= \varepsilon(h)1\otimes m.\end{aligned}$$

(2)$\Rightarrow$(1); By (2), there is an H-collinear and invertible map $J: H\to E$. Let J' be $*$-inverse to J. Define $q: A\otimes_B M\to M$ by the equation

$$\Sigma\, J'(a_1)(a_0\otimes m) = 1\otimes q(a\otimes m) \qquad \text{in } A\otimes_B M.$$

Then define $\Phi: A\otimes_B M\to M\otimes H$ and $\Psi: M\otimes H\to A\otimes_B M$ by $\Phi(a\otimes m) := \Sigma\, q(a_0\otimes m)\otimes a_1$, $\Psi(m\otimes h) := J(h)(1\otimes m)$. Since J is collinear, for all $h\in H$, $\Delta_E(J'(h)) = \Sigma\, J'(h_2)\otimes S(h_1)$ (cf. [9], 3.2). Hence $\Sigma\, J'(a_1)(a_0\otimes m)$ is an H-coinvariant element of $N = A\otimes_B M$, since

$$\Delta_N(\Sigma\,(a_0\otimes m)\cdot J'(a_1)) = \Sigma\,(a_0\otimes m)\cdot J'(a_3)\otimes a_1S(a_2) = \Sigma\,(a_0\otimes m)\cdot J'(a_1)\otimes 1.$$

Therefore, by 1.3(5), q is well-defined and B-linear. Now it is easily checked that Φ and Ψ are inverse isomorphisms.

The constructions in the preceding proof are similar to [21], 2.1, 2.2. As the referee pointed out, the implication (2)$\Rightarrow$(1) in 3.6 can also be derived from [9], 3.6 (applied to $A := H$, $B := E$), since $A\otimes_B M\in\mathcal{M}_E^H$ by 3.1, and $(A\otimes_B M)^{\mathrm{co}\,H}\cong M$ by 1.3(5).

The next corollary shows that stability over a pointed Hopf algebra H can be split into two conditions (a) and (b): If H is irreducible, then (b) trivially holds. If H is a group algebra, then (a) is always true. 3.7 answers a question of Takeuchi.

3.7. COROLLARY. *Assume situation (S). If k is a field and H is pointed, then the following are equivalent*:

(1) *M is stable.*

(2) (a) *$M\to A\otimes_B M$, $m\mapsto 1\otimes m$, is a B-split monomorphism.*

(b) *For all group-like elements g of H, $M\cong A_g\otimes_B M$ as left B-modules.*

PROOF. (1)$\Rightarrow$(2): By (1) and 3.2(5), there is a left B-linear and right H-collinear isomorphism $\Phi: A \otimes_B M \to M \otimes H$ such that $\Phi(1 \otimes m) = m \otimes 1$ for all $m \in M$. In particular, (a) holds, since $M \to M \otimes H$, $m \mapsto m \otimes 1$, is a B-split monomorphism. By 1.3(3), $\Phi \square_H kg$ induces the isomorphism $A_g \otimes_B M \to M \otimes kg$. This proves (b).

(2)$\Rightarrow$(1): Let $H_0 = k[G]$ be the coradical of H, G the set of group-like elements of H. Define $A_0 = A(H_0)$. Then, by [22], 3.11(2), $B \subset A_0$ is a right H_0-Galois extension and A_0 is faithfully flat as right B-module. The direct sum of the isomorphisms in (b) shows that M is A_0-stable. Hence, by 3.6, there is a right H_0-collinear and invertible map $J_0: H_0 \to E_0 := \mathrm{End}_{A_0}(A_0 \otimes_B M)^{\mathrm{op}}$. By (a) and 3.3, E is an injective H-comodule (note that the antipode of H is bijective since H is pointed). Therefore, the H-collinear map $H_0 \xrightarrow{J_0} E_0 \subset E$ can be lifted to an H-collinear map $J: H \to E$. Then J is invertible, since $J \mid H_0 = J_0$ is invertible. Hence, by 3.6, M is A-stable.

3.8. REMARK. Assume situation (S). One can check that the constructions in the proof of 3.6 give a bijection between

(a) all B-linear maps $q: A \otimes_B M \to M$ which define an A-module structure on M extending the given B-module structure on M, and

(b) all k-linear maps $J': H \to E$ such that $\Delta_E(J'(x)) = \Sigma\, J'(x_2) \otimes S(x_1)$ and $J'(xy) = J'(y)J'(x)$, $J'(1) = 1$ for all $x, y \in H$.

Therefore, if the antipode of H is bijective, there is an A-module structure on M extending the given B-module structure if and only if $E' \subset E$ is a trivial crossed product, i.e. there is a right H-collinear algebra map $J: H \to E$ (define $J := J'S^{-1}$).

4. The stabilizer

Let k again be a field. If R is an algebra and T a coalgebra, then ${}_R\mathcal{M}\mathrm{od}^T$ will denote the category of (R, T)-bimodules X, i.e. of right T-comodules which are left R-modules such that the comodule structure map Δ_X is R-linear, that is, $\Delta_X(rx) = \Sigma\, rx_0 \otimes x_1$ for all $r \in R$ and $x \in X$. If M is a left R-module, and $S \subset T$ a subcoalgebra, then $M \otimes S$ is an (R, T)-bimodule in the natural way ($r(m \otimes s) := rm \otimes s$ and $\Delta_{M \otimes S}(m \otimes s) = m \otimes \Delta(s)$ define the module and comodule structure). If $f: S \to T$ is a coalgebra map, and $X \in {}_R\mathcal{M}\mathrm{od}^T$, then $X \square_T S \in {}_R\mathcal{M}\mathrm{od}^S$ as submodule and subcomodule of $X \otimes S$ (cf. [22]).

The stabilizer as a coalgebra exists in the general context of (R, T)-bimodules. This is shown in the next theorem generalizing [12].

4.1. THEOREM. *Let R be an algebra, T a coalgebra, M a left R-module and $X \in {}_R\mathcal{M}\mathrm{od}^T$.*

(1) *Let $S \subset T$ be a subcoalgebra and $\Phi : M \otimes T \to X$ an isomorphism in ${}_R\mathcal{M}\mathrm{od}^T$.*

(a) *Then $\Phi_S : M \otimes S \cong M \otimes T \square_T S \xrightarrow{\Phi \square S} X \square_T S$ is an isomorphism in ${}_R\mathcal{M}\mathrm{od}^S$, where the first map is the canonical isomorphism defined by the diagonal of S.*

(b) *If $\Psi : M \otimes S \to X \square_T S$ is any isomorphism in ${}_R\mathcal{M}\mathrm{od}^S$, then there is an isomorphism $\Phi' : M \otimes T \to X$ in ${}_R\mathcal{M}\mathrm{od}^T$ such that $\Psi = \Phi'_S$ (defined as in* (a)).

(2) *Assume X is injective as right T-comodule. Then the sum of all subcoalgebras $S \subset T$ such that $M \otimes S \cong X \square_T S$ in ${}_R\mathcal{M}\mathrm{od}^S$ is a subcoalgebra of T having the same property.*

PROOF. (1)(a) is obvious. To prove (1)(b), write $T_0 = S_0 \oplus C$, where S_0 and T_0 are the coradicals of S and T, and C is the sum of the simple subcoalgebras of T not contained in S. Then $D := S + C = S \oplus C$, and the direct sum of Ψ and $\Phi \square_T C$ defines an isomorphism $\Lambda : M \otimes D \to X \square_T D$ in ${}_R\mathcal{M}\mathrm{od}^T$. Now $X \cong M \otimes T$ is injective relative to R-split monomorphisms in ${}_R\mathcal{M}\mathrm{od}^T$, since R-linear and T-collinear maps into $M \otimes T$ are given by R-linear maps into M. Hence there is a map Φ' of (R, T)-bimodules such that the following diagram of (R, T)-bimodules commutes:

$$\begin{array}{ccc} M \otimes D & \xrightarrow{\Lambda} & X \square_T D \\ \downarrow & & \downarrow \\ M \otimes T & \xrightarrow{\Phi'} & X \end{array}$$

where the vertical maps are the canonical inclusion maps. Note that the inclusion $M \otimes D \to M \otimes T$ is an R-split monomorphism of (R, T)-bimodules. By the commutativity of the diagram, the composition

$$M \otimes D \cong M \otimes T \square_T D \xrightarrow{\Phi' \square D} X \square_T D$$

is the isomorphism Λ. In particular, since $S \subset D$ and $T_0 \subset D$, $\Phi'_S = \Psi$, and $\Phi' \square_T T_0$ is identified with the isomorphism $\Lambda \square_D T_0$. Hence, Φ' is bijective, since $M \otimes T$ is an injective T-comodule (cf. [22], 1.2(1)).

(2) (a) Let C and D be subcoalgebras of T, and $\Phi : M \otimes C \cong X \square_T C$ in ${}_R\mathcal{M}\mathrm{od}^C$, $\Psi : M \otimes D \cong X \square_T D$ in ${}_R\mathcal{M}\mathrm{od}^D$. Then $M \otimes (C + D) \cong X \square_T (C + D)$ in ${}_R\mathcal{M}\mathrm{od}^{C+D}$.

PROOF. By (1), one can assume that $\Phi_{C\cap D} = \Psi_{C\cap D}$ (replace Ψ by an extension of $\Phi_{C\cap D}$). The usual exact sequence $0 \to C \cap D \xrightarrow{i} C \times D \xrightarrow{p} C + D \to 0$, $i(t) = (t, -t)$, $p(c, d) = c + d$, is exact in the category of left T-comodules. Hence,

$$0 \to M \otimes (C \cap D) \to (M \otimes C) \oplus (M \otimes D) \to M \otimes (C + D) \to 0$$

and

$$0 \to X \square_T (C \cap D) \to (X \square_T C) \oplus (X \square_T D) \to X \square_T (C + D) \to 0$$

are exact, since X is coflat (= injective) as right T-comodule. The isomorphism $\Phi \oplus \Psi$ between the terms in the middle induces an isomorphism of the kernels, since $\Phi_{C\cap D} = \Psi_{C\cap D}$. Therefore, $\Phi \oplus \Psi$ induces an isomorphism between $M \otimes (C + D)$ and $X \square_T (C + D)$ in ${}_R\mathcal{M}\mathrm{od}^{C+D}$.

(b) Consider the set of all pairs (C, Φ), where $C \subset T$ is a subcoalgebra, and $\Phi : M \otimes C \to X \square_T C$ is an isomorphism in ${}_R\mathcal{M}\mathrm{od}^C$. If (C, Φ) and (D, Ψ) are two such pairs, define $(C, \Phi) \leqq (D, \Psi)$, if C is contained in D and Φ is the restriction of Ψ. By Zorn's lemma, there is a maximal pair (S, Φ). If (D, Ψ) is any pair, then there is an isomorphism $\Phi' : M \otimes (S + D) \cong X \square_T (S + D)$ in ${}_R\mathcal{M}\mathrm{od}^{S+D}$ by (a), and, by (1) (replace X by $X \square_T (S + D)$), one can assume that $(S, \Phi) \leqq (S + D, \Phi')$. Hence, $D \subset S$ by the maximality of (S, Φ).

The following abstract theorem on (R, T)-bimodules is the main technical device to show that the stabilizer is a bialgebra. The proof generalizes ideas of Greipel [12] and uses a simplification due to Takeuchi.

4.2. THEOREM. *Let R be an algebra, S and T coalgebras, and $f : S \to T$ a surjective coalgebra map. Let $X \in {}_R\mathcal{M}\mathrm{od}^T$, and assume X is injective as right T-comodule. Let M be a left R-module. Assume one of the following conditions:*

(a) *S is pointed.*

(b) *There is a field extension $k \subset k'$ such that $S \otimes k'$ is pointed, and M is an R-module of finite length.*

If $M \otimes S \cong X \square_T S$ in ${}_R\mathcal{M}\mathrm{od}^S$, then $M \otimes T \cong X$ in ${}_R\mathcal{M}\mathrm{od}^T$ (here, S is a T-comodule via f).

PROOF. By 4.1, it suffices to show that $M \otimes T' \cong X \square_T T'$ in ${}_R\mathcal{M}\mathrm{od}^{T'}$ for all finite dimensional subcoalgebras $T' \subset T$. Therefore, one can assume that S is finite dimensional, hence pointed after some finite field extension. Then it is enough to consider only the case when S is pointed. This follows from the theorem of Deuring–Noether. Assume (b). $M \otimes T'$ and $X \square_T T'$ are (R, T')-

bimodules or equivalently left $R \otimes T'^*$-modules, and $M \otimes T'$ has finite length over $R \otimes T'^*$. Therefore, the $R \otimes T'^*$-modules $M \otimes T'$ and $X \square_T T'$ are already isomorphic if they become isomorphic after some finite field extension. The proof of 4.2 (under the assumption that S is finite dimensional and pointed) proceeds by induction on the dimension of S.

Take a maximal subcoalgebra $C \subset S$. Then C has codimension 1 in S (the minimal ideals in S^* are 1-dimensional, since $S^*/\mathrm{Ra}(S^*) \cong k^n$ for some n). If $f(C) = T$, then $M \otimes T \cong X$ by induction hypothesis. If $D := f(C) \neq T$, then the surjective map f induces an isomorphism $S/C \to T/D$, since S/C is 1-dimensional. Therefore, the kernel of f lies in C. Let I be the kernel of f. By induction, there is an isomorphism $\Phi : M \otimes D \to X \square_T D$ in ${}_R\mathcal{M}\mathrm{od}^D$. Define Ψ as the composition

$$M \otimes C \cong M \otimes D \square_D C \xrightarrow{\Phi \square C} X \square_T D \square_D C \cong X \square_T C.$$

Since $M \otimes S \cong X \square_T S$, by 4.1, there is an isomorphism $\Psi' : M \otimes S \cong X \square_T S$ in ${}_R\mathcal{M}\mathrm{od}^S$ such that $\Psi = \Psi'_C$.

Consider the exact sequences in ${}_R\mathcal{M}\mathrm{od}^T$

$$0 \to M \otimes I \to M \otimes S \xrightarrow{M \otimes f} M \otimes T \to 0,$$

$$0 \to X \square_T I \to X \square_T S \to X \square_T T \cong X \to 0.$$

The lower sequence is exact, since X is coflat as T-comodule. It suffices to show that the isomorphism Ψ' between the terms in the middle induces an isomorphism of the kernels. Take elements $m \in M$ and $s \in I$. Since $\Psi'_C = \Psi$ and $I \subset C$,

$$\Psi'(m \otimes s) = \Sigma\, (1 \otimes \varepsilon)\Phi(m \otimes f(s_1)) \otimes s_2.$$

Hence, $(1 \otimes f)\Psi'(m \otimes s) = 0$, since $\Sigma\, f(s_1) \otimes s_2 \in T \otimes I$. Therefore, Ψ' maps $M \otimes I$ into $X \square_T I$. Similarly, one shows that the inverse of Ψ' maps $X \square_T I$ into $M \otimes I$.

Now consider again the H-Galois extension $B \subset A$ of Section 1.

Definition. Let M be a left B-module.

(1) Let $C \subset H$ be a subcoalgebra. C is said to *stabilize* M if

$$M \otimes C \cong A(C) \otimes_B M \text{ in } {}_B\mathcal{M}\mathrm{od}^C.$$

(2) Let $\mathrm{St}(M, H)$ be the sum of all subcoalgebras $C \subset H$ which stabilize M. $H_{st} := \mathrm{St}(M, H)$ is called the *stabilizer of M in H*.

The next lemma generalizes the isomorphism $A_g \otimes_B A_h \cong A_{gh}$ of 1.5.

4.3. LEMMA. *Assume the situation of the definition. Let C and D be subcoalgebras of H. Consider $C \otimes D$ as left H-comodule via the coalgebra map $C \otimes D \to H$ defined by mutliplication. Then*

$$A(C) \otimes_B A(D) \to A \,\square_H (C \otimes D), \quad x \otimes y \mapsto \Sigma\, x_0 y_0 \otimes x_1 \otimes y_1,$$

is a left and right B-linear isomorphism.

PROOF. Let Φ be the above described map. Obviously, Φ is well-defined and B-linear, where $A \,\square_H (C \otimes D)$ is a B-bimodule by multiplication on A. Since A_B is faithfully flat, it is enough to show $A \otimes_B \Phi$ is bijective.

For any left H-comodule V, the map

$$A \otimes_B (A \,\square_H V) \cong (A \otimes_B A) \,\square_H V \cong A \otimes H \,\square_H V \cong A \otimes V,$$

$$\Sigma\, x \otimes y_i \otimes v_i \mapsto \Sigma\, x y_i \otimes v_i,$$

is an isomorphism, since A_B is flat. Applying this map twice, first to $V = C$ and then to $V = D$, gives an isomorphism $A \otimes_B (A \,\square_H C) \otimes_B (A \,\square_H D) \to A \otimes C \otimes D$. This isomorphism can be identified with $A \otimes_B \Phi$.

4.4. THEOREM. *Let M be a left B-module and H_{st} the stabilizer of M in H.*

(1) *H_{st} is a subcoalgebra of H, and any subcoalgebra of H_{st} stabilizes M.*

(2) *Assume one of the following conditions:*

 (a) *H is pointed.*

 (b) *There is a field extension $k \subset k'$ such that $H \otimes k'$ is pointed, and M is a B-module of finite length.*

 Then H_{st} is a Hopf subalgebra of H.

PROOF. (1) follows from 4.1 and 1.3(3), where $R := B$, $T := H$ and $X := A \otimes_B M$.

(2)(i) Define $C := H_{st}$. By 4.3,

$$A(C) \otimes_B A(C) \cong A \,\square_H (C \otimes C), \quad x \otimes y \mapsto \Sigma\, x_0 y_0 \otimes x_1 \otimes y_1,$$

is a left and right B-linear isomorphism. Hence, by tensoring with M one obtains

$$\begin{aligned}(A\otimes_B M)\,\square_H\,(C\otimes C) &\cong (A\,\square_H\,(C\otimes C))\otimes_B M && \text{(by 1.3)}\\ &\cong A(C)\otimes_B A(C)\otimes_B M && \text{(by 4.3)}\\ &\cong A(C)\otimes_B M\otimes C && \text{(since } C \text{ stabilizes } M)\\ &\cong M\otimes C\otimes C && \text{(since } C \text{ stabilizes } M).\end{aligned}$$

It can be checked that this isomorphism is a morphism in ${}_B\mathcal{M}\mathrm{od}^{C\otimes C}$.

Now apply 4.2 where $R := B$, $S := C\otimes C$ and $T := C\cdot C$ as subcoalgebra of H, f is the multiplication map and $X := A(T)\otimes_B M$. By 1.3(3), $X \cong (A\otimes_B M)\,\square_H\,T$ is injective as right T-comodule. Then, by 4.2, $M\otimes T \cong A(T)\otimes_B M$. By definition of the stabilizer, this means $T = C\cdot C$ is contained in $H_{\mathrm{st}} = C$. Thus the stabilizer is a subbialgebra of H.

(ii) The subbialgebra H_{st} of H is a Hopf subalgebra if the antipode S of H maps the coradical of H_{st} into H_{st}. This means: For any finite dimensional subcoalgebra C of the coradical of H_{st} there is an isomorphism $A(S(C))\otimes_B M \cong M\otimes S(C)$ in ${}_B\mathcal{M}\mathrm{od}^H$. Take any such C. By assumption, there is a finite field extension $k' \subset k$ such that $C\otimes k'$ is pointed. Hence, by Deuring–Noether as in the proof of 4.2, one can assume that C is pointed. By 4.1 and 1.3(3), it is sufficient to consider the case $C = kg$, g a group-like element in H_{st}.

Any group-like element h of H lies in the stabilizer if and only if $M \cong A_h\otimes_B M$ as left B-modules. Therefore, $g\in H_{\mathrm{st}}$ implies $M\cong A_g\otimes_B M$ and $A_{g^{-1}}\otimes_B M \cong A_{g^{-1}}\otimes_B A_g\otimes_B M \cong M$, by 1.5(1). Thus, g^{-1} lies in the stabilizer.

4.5. Lemma. *Under the assumptions of* 4.4, *let $D \subset C$ be subcoalgebras of the stabilizer H_{st} of M. Then $A(D)\otimes_B M \to A(C)\otimes_B M$, $a\otimes m \mapsto a\otimes m$, is a B-split monomorphism.*

Proof. Since C is contained in the stabilizer, there is an isomorphism $\Phi : A(C)\otimes_B M \cong M\otimes C$ in ${}_B\mathcal{M}\mathrm{od}^C$. By 1.3(3),

$$(A(C)\otimes_B M)\,\square_C\,D \cong (A\otimes_B M)\,\square_H\,C\,\square_C\,D \cong A(D)\otimes_B M.$$

Hence, restriction of Φ is an isomorphism $A(D)\otimes_B M \to M\otimes D$. Therefore, $A(D)\otimes_B M \to A(C)\otimes_B M$ is a B-split monomorphism.

Finally, a useful criterion for a coalgebra to lie in the stabilizer will be given in terms of the splitting condition in 4.5.

4.6. Theorem. *In the situation of* 4.4, *let $C \subset H$ be a subcoalgebra. Then C is contained in the stabilizer H_{st} of M if and only if*

(a) *the coradical C_0 of C is contained in H_{st}, and*

(b) *the inclusion map $A(C_0)\otimes_B M \to A(C)\otimes_B M$ is a B-split monomorphism.*

PROOF. If C lies in the stabilizer, then (a) and (b) follow from 4.4(1) and 4.5. Conversely, assume (a) and (b). By (a), there is an isomorphism $\Phi_0 : A(C_0)\otimes_B M \to M\otimes C_0$ in ${}_B\mathcal{M}od^H$. By (b), there is a B-linear map $f: A(C)\otimes_B M \to M$ such that $f\,|\,A(C_0)\otimes_B M$ is $(1\otimes\varepsilon)\Phi_0$. Define

$$\Phi : A(C)\otimes_B M \to M\otimes C \quad \text{by} \quad \Phi(a\otimes m) := \Sigma\, f(a_0\otimes m)\otimes a_1.$$

By restriction, Φ induces the isomorphism Φ_0. But, by 1.3(3), $(A(C)\otimes_B M)\,\square_C\, C_0 \cong A(C_0)\otimes_B M$. Thus, $\Phi\,\square_C\, C_0$ is bijective. Therefore, Φ is bijective by [22], 1.2(1), since $A(C)\otimes_B M$ is an injective C-comodule by 1.3(3). Hence, by definition, $C \subset H_{st}$.

4.7. REMARK. (1) If C is pointed, then condition (a) in 4.6 is equivalent to

(a_1) For all group-like elements g in C, $A_g\otimes_B M \cong M$ as B-modules.

(2) In the situation of 4.6, let C be a pointed Hopf subalgebra of H. Assume C is finite dimensional or M is a B-module of finite presentation. Then C lies in the stabilizer H_{st} of M if and only if the following conditions hold:

(a_1) in (1).

(b_1) The inclusion map $M \to A(C)\otimes_B M$, $m \mapsto 1\otimes m$, is a B-split monomorphism.

PROOF. This is a restatement of 3.7. C is contained in the stabilizer if and only if there is an isomorphism $A(C)\otimes_B M \cong M\otimes C$ in ${}_B\mathcal{M}od^C$. But, by [22], 3.11(2), $B \subset A(C)$ is a C-Galois extension, and $A(C)$ is faithfully flat as right B-module. Hence, by 3.7, M is $A(C)$-stable if and only if (a_1) and (b_1) hold.

4.8. REMARK. (1) Assume the antipode of H is bijective and $B \subset A$ is an H-crossed product, i.e. there is a right H-collinear and invertible map $j: H \to A$. Let j' be $*$-inverse to j, and let $H\otimes B \to B$, $h\otimes b \mapsto h(b) := \Sigma\, j(h_1)bj'(h_2)$, be the induced action of H on B. Then

$$h(bb') = \Sigma\, h_1(b)h_2(b'), \qquad h(1) = \varepsilon(h)1,$$

but in general this action does not define an H-module structure on B.

Let $C \subset H$ be a subcoalgebra. Then the induced module can be described as follows:

$$A(C)\otimes_B M \cong (M\otimes C) \quad \text{in } {}_B\mathcal{M}od^C,$$

where $(M \otimes C)$ is $M \otimes C$ as k-module and C-comodule, and where the B-module structure $\circ$ is twisted: $b \circ (m \otimes h) := \Sigma\, (S^{-1}(h_1))(b)m \otimes h_2$.

This follows from the isomorphism $B \otimes H \to A$, $b \otimes h \mapsto j'(S^{-1}(h))b$. (The inverse map is given by $a \mapsto \Sigma\, j(S^{-1}(a_1))a_0 \otimes a_2$.)

Let $(C \otimes M)$ be $C \otimes M$ as left-comodule with the following twisted B-module structure $\circ: b \circ (h \otimes m) := \Sigma\, h_1 \otimes h_2(b)m$. Then C stabilizes M if and only

$$C \otimes M \cong (C \otimes M) \quad \text{in } {}^{C}_{B}\mathcal{M}.$$

This is clear from the above description of the induced module, and for the following reason: Any isomorphism $(M \otimes C) \to M \otimes C$, $m \otimes h \mapsto \Sigma\, \varphi(m \otimes h_1) \otimes h_2$, induces the isomorphism $(C \otimes M) \to C \otimes M$, $h \otimes m \mapsto \Sigma\, h_1 \otimes \varphi(m \otimes S(h_2))$, and conversely.

(2) Assume now that A is a cocommutative Hopf algebra, B a normal Hopf subalgebra and $H = \bar{A} := A/AB^+$ the quotient Hopf algebra as in example (4) of the introduction. Assume that $B \subset A$ is an H-crossed product as in (1) (this holds, for example, if A is pointed or B is finite dimensional, cf. [22], Th. III).

Let $C \subset H$ be a subcoalgebra which stabilizes M. Then $D := A(C)$ is a subcoalgebra of A and

$$_{\lambda}(D \otimes M) \cong D \otimes M \quad \text{in } {}^{D}_{B}\mathcal{M}\text{od},$$

where $_{\lambda}(D \otimes M)$ is $D \otimes M$ as left D-comodule with the following twisted B-module operation

$$\underset{\lambda}{\circ} : b \underset{\lambda}{\circ} (a \otimes m) := \Sigma\, a_1 \otimes \lambda(a_2 \otimes b)m, \quad \lambda(a \otimes b) := \Sigma\, a_1 b S(a_2).$$

Since B is normal in A, $\lambda : A \otimes B \to B$ is well-defined.

Proof. By (1), $(C \otimes M) \cong C \otimes M$. Cotensoring with $A \square_H$- yields an isomorphism $(D \otimes M) \cong D \otimes M$ in ${}^{D}_{B}\mathcal{M}$od. Here, the twisted B-module structure on $(D \otimes M)$ is $b \circ (a \otimes m) := \Sigma\, a_1 \otimes \overline{a_2}(b)m$. $D = A(C) \cong A \square_H C$ is a subcoalgebra since A is cocommutative. It remains to show that $_{\lambda}(D \otimes M) \cong (D \otimes M)$.

The map $f: A \to B$, $f(a) := \Sigma\, j(\overline{a_1})S(a_2)$, is well-defined, since $B = A^{\mathrm{co}\, H}$. Then $_{\lambda}(D \otimes M) \to (D \otimes M)$, $a \otimes m \mapsto \Sigma\, a_1 \otimes f(a_2)m$, is an isomorphism in ${}^{D}_{B}\mathcal{M}$od.

(3) In the situation of (2), Greipel [12] defined the stabilizer A_{st} of M as the sum of all subcoalgebras $D \subset A$ such that $_{\lambda}(D \otimes M) \cong D \otimes M$ in ${}^{D}_{B}\mathcal{M}$od.

(He used λ' instead of λ, where $\lambda'(a \otimes b) := \Sigma\, S(a_1)ba_2$. Note that ${}_\lambda(D \otimes M) \cong {}_{\lambda'}(D \otimes M)$, $a \otimes m \mapsto S(a) \otimes m$.)

To prove that A_{st} is a Hopf subalgebra of A, he used the fact that λ defines an A-left module structure on B. Since the above operation of H on B in general does not define an H-module structure, Greipel's theory cannot be applied to the more general situation of the H-crossed product in (1).

(4) If H is a group algebra $k[G]$, then clearly $H_{st} = k[G_{st}]$, where G_{st} is the stabilizer as defined by Dade [5], cf. [4], p. 269, for strongly graded algebras:

$$G_{st} = \{g \in G \mid A_g \otimes_B M \cong M \text{ over } B\}.$$

If Γ' is a normal subgroup of a group Γ and $G := \Gamma/\Gamma'$, $B := k[\Gamma'] \subset A := k[\Gamma]$, then G_{st} is the quotient group Γ_{st}/Γ', where Γ_{st} is the usual stabilizer

$$\Gamma_{st} := \{\gamma \in \Gamma \mid {}_\gamma M \cong M \text{ over } k[\Gamma']\}.$$

(5) Let k be a field of characteristic 0, $\mathfrak{g}$ a Lie algebra and $\mathfrak{g}' \subset \mathfrak{g}$ a Lie ideal. Define $B := U\mathfrak{g}' \subset A := U\mathfrak{g}$ and $H := U\mathfrak{g}/\mathfrak{g}'$. Then $H_{st} = U\mathfrak{g}_{st}/\mathfrak{g}'$, where $\mathfrak{g}_{st}$ is the stabilizer of M in the sense of Blattner [2], cf. [8], 5.3.

(6) In the situation of example (3) of the introduction, let M be a finite dimensional left G'-module. Equivalently, M is a left B-module. Let G_{st} be the stabilizer of M as defined by Voigt in [28], 1.3. Then one can show that $A(H_{st}) = H(G_{st})$, in other words, H_{st} is the cocommutative Hopf algebra of the quotient group scheme G/G_{st}. More generally, in another paper the stabilizer for cocommutative Hopf algebras will be characterized functorially.

5. Induction of simple modules

In this section, Blattner's irreducibility criterion [2] will be proved in the context of Hopf Galois extensions generalizing results of Voigt [28] and Greipel [12].

5.1. Lemma. *Assume H is pointed. Let M be a simple left B-module and $C \subset H$ a subcoalgebra such that $A(C) \otimes_B M$ is M-isotypically semisimple as left B-module. Then C is contained in the stabilizer of M in H.*

Proof. Let g be a group-like element of C. Then $A_g \otimes_B M$ is a B-submodule of $A(C) \otimes_B M$. Hence $A_g \otimes_B M$ is isomorphic to a direct sum of copies of M. But, by 1.5(2), $A_g \otimes_B M$ is simple. Thus, $A_g \otimes_B M \cong M$. Furthermore, $A(C_0) \otimes_B M$ is a B-submodule of $A(C) \otimes_B M$, hence a B-direct summand, since

$A(C)\otimes_B M$ is semisimple by assumption. Therefore, both conditions (a) and (b) in 4.6 are satisfied, and C lies in the stabilizer of M.

5.2. Corollary. *Assume H is pointed.*

(1) *Let M be a simple left B-module and X a left B-module which is M-isotypically semisimple. Then M and X have the same stabilizer in H.*

(2) *Let N be a left A-module, M a simple B-submodule and X the M-homogeneous component of the B-socle of N. Let H_{st} be the stabilizer of M in H. Then X is an $A(H_{st})$-submodule of N.*

Proof. (1) Let $C \subset H$ be a subcoalgebra. If C stabilizes M, then clearly C stabilizes $X \cong M^{(I)}$, I some set. Conversely, if C stabilizes X, then $A(C)\otimes_B M$ is a B-submodule of $A(C)\otimes_B X \cong X \otimes C$ which is M-isotypically semisimple. Hence, by 5.1, C stabilizes M.

(2) By (1), $A(H_{st})\otimes_B X \cong X \otimes H_{st}$ is M-isotypically semisimple as left B-module. Therefore, the image of the multiplication map $A(H_{st})\otimes_B X \to N$ lies in X, the M-homogeneous component of the socle of N.

5.3. Lemma. *Let M be a simple B-module, g a group-like element of H and V a k-module. Assume that k is the center of $\Gamma := \mathrm{End}_B(M)$. Consider $A_g \otimes_B M \otimes V$ as left $B\otimes\Gamma$-module in the obvious way, where $(b\otimes\gamma)(a\otimes m\otimes v) := ba\otimes\gamma(m)\otimes v$. Then any $B\otimes\Gamma$-submodule of $A_g\otimes_B M\otimes V$ is of the form $A_g\otimes_B M\otimes U$, U a submodule of V.*

Proof. In general, if Z is any simple left B-module and $D := \mathrm{End}_B(Z)$, then $V \mapsto Z\otimes_{D^{op}} V$ is an equivalence between the category of left D^{op}-modules and the category of left B-modules which are Z-isotypically semisimple. Now assume k is the center of D. Then Z is a simple left $B\otimes D$-module with endomorphism ring k. Hence, $V \mapsto Z\otimes V$ is an equivalence between the category of k-modules and the category of left $B\otimes D$-modules which are Z-isotypically semisimple. By 1.5(2), $A_g\otimes_B M$ is a simple B-module with endomorphism ring isomorphic to Γ. Therefore, the previous remark in case $Z := A_g\otimes_B M$ proves the lemma.

In the next theorem, which is a very general form of Blattner's theorem [2], Th. 3, the following notation will be used: If V is a left B-module and M a simple left B-module, let $\mathrm{So}(V)$ be the B-socle of V and M-$\mathrm{So}(V)$ the M-homogeneous component of $\mathrm{So}(V)$.

5.4. Theorem. *Assume H is cocommutative and pointed. Let G be the set of group-like elements of H and $H_0 = k[G]$ its coradical. Let M be a simple left*

B-module such that k is the center of its B-endomorphism ring. Let H_{st} be the stabilizer of M in H. Define $S := A(H_{st})$ and $A_0 := A(H_0)$.

Then $A(H_0H_{st}) = A_0S = \Sigma_g A_gS$, and

(1) (a) *For all $g \in G$: $A_g \otimes_B M$-$\mathrm{So}(A \otimes_B M) \cong A_gS \otimes_B M$.*

(b) $\mathrm{So}(A \otimes_B M) \cong A_0S \otimes_B M$.

(2) *If Q is a left S-module such that Q is M-isotypically semisimple as B-module (by restriction), then:*

(a) *For all $g \in G$: $A_g \otimes_B M$-$\mathrm{So}(A \otimes_S Q) \cong A_gS \otimes_S Q$.*

(b) $\mathrm{So}(A \otimes_S Q) \cong A_0S \otimes_S Q$.

(All isomorphisms are induced by the inclusions $A_gS \subset A$ and $A_0S \subset A$.)

Proof. (A) First it will be shown that M-$\mathrm{So}(A \otimes_B M) \cong S \otimes_B M$.

Let X be the M-homogeneous component of the B-socle of $A \otimes_B M$. Then $S \otimes_B M \subset X \subset A \otimes_B M$, since $S \otimes_B M \cong M \otimes H_{st}$ is B-isomorphic to a direct sum of copies of M. Let $\Gamma := \mathrm{End}_B(M)$ be the endomorphism ring of M. Consider $A \otimes_B M$ as left $B \otimes \Gamma$-module with module structure $(b \otimes \gamma)(a \otimes m) := ba \otimes \gamma(m)$. Then X is a $B \otimes \Gamma$-submodule of $A \otimes_B M$.

Since the comodule structure map $A \otimes_B M \to A \otimes_B M \otimes H$ is left B-linear, X is also a right H-subcomodule of $A \otimes_B M$. Define $Y := X/(S \otimes_B M)$, a left $B \otimes \Gamma$-submodule and a right H-subcomodule of $(A \otimes_B M)/(S \otimes_B M) \cong A/S \otimes_B M$.

It will suffice to show $Y_g = 0$ for all group-like elements g of H, since this implies $Y_0 = 0$, hence $Y = 0$, or equivalently $X = S \otimes_B M$.

Let g be any group-like element of H. Then Y_g is a $B \otimes \Gamma$-submodule of $(A/S \otimes_B M)_g$. By 1.2 and 1.4(3), the canonical maps

$$(A/S \otimes_B M)_g \leftarrow (A/S)_g \otimes_B M \rightarrow A_g \otimes_B M \otimes (H/H_{st})_g$$

are bijective. Both maps are clearly $B \otimes \Gamma$-linear. Hence, by 5.3, Y_g is $B \otimes \Gamma$-isomorphic to $A_g \otimes_B M \otimes C/H_{st}$, where C is a k-submodule of H containing H_{st} such that $C/H_{st} = (H/H_{st})_g$.

Take any $c \in C$. Then $\Delta(c) \in c \otimes g + H_{st} \otimes H \subset C \otimes H$. Therefore, $\Delta(C) \subset C \otimes H$. Hence C is a subcoalgebra of H, since H is cocommutative.

Then, by 1.4(1) and (2), the canonical map

$$A(C)/S \otimes_B M \rightarrow A_g \otimes_B M \otimes C/H_{st} \text{ is bijective.}$$

(Note that ${}_g(C/H_{st}) = (C/H_{st})_g = C/H_{st}$, since C is cocommutative.)

This means that $Y_g = (A(C) \otimes_B M)/(S \otimes_B M)$ is a submodule of $Y = X/(S \otimes_B M)$. Hence $A(C) \otimes_B M$ is a submodule of X. Then, by the definition of

X, $A(C)\otimes_B M$ is M-isotypically semisimple, and, by 5.1, C is contained in the stabilizer H_{st}, or equivalently, Y_g is zero.

(B) Proof of (1)(a): Let $g\in G$. By 1.5(2), $A_g\otimes_B M$ is simple over B. Let

$$Z := M\text{-So}(A\otimes_B M) \quad \text{and} \quad Z_g := A_g\otimes_B M\text{-So}(A\otimes_B M).$$

It follows easily from 1.5(1) that Z_g is the image of the multiplication map $A_g\otimes_B Z\to A\otimes_B M$, $a\otimes z\mapsto az$. By part (A), $Z\cong S\otimes_B M$. Hence, $Z_g\cong A_g(S\otimes_B M)=A(gH_{\mathrm{st}})\otimes_B M$, since $A_gA(H_{\mathrm{st}})=A(gH_{\mathrm{st}})$ by 1.5(3).

(C) Proof of (1)(b): By 2.1, any simple B-submodule of $A\otimes_B M$ is isomorphic to $A_g\otimes_B M$ for some g. Hence, by part (B), $\mathrm{So}(A\otimes_B M)$ is the sum of all submodules $A_gS\otimes_B M$. But this sum is equal to $A_0S\otimes_B M$, since

$$A(H_0H_{\mathrm{st}})=A(\Sigma_g gH_{\mathrm{st}})=\Sigma_g A(gH_{\mathrm{st}})=\Sigma_g A_gS=A_0S$$

by 1.5(4).

(D) Proof of (2)(b): Since H is cocommutative, H is faithfully flat over H_{st} (cf. 1.1(4)). By 1.7(1), the inclusion induces an injective map $A_0S\otimes_S Q\to A\otimes_S Q$, since $A_0S=A(H_0H_{\mathrm{st}})$. Let $\overline{H}=H/HH_{\mathrm{st}}^+$.

By 1.6, $A\otimes_B A\to(A\otimes_S Q)\,\square_{\overline{H}}\,H$ is a left B-linear isomorphism. In particular, $\mathrm{So}(A\otimes_B Q)$ is mapped isomorphically onto $\mathrm{So}((A\otimes_S Q)\,\square_{\overline{H}}\,H)\cong\mathrm{So}(A\otimes_S Q)\,\square_{\overline{H}}\,H$ (note that $\mathrm{So}(A\otimes_S Q)$ is an $\overline{H}$-subcomodule of $A\otimes_S Q$).

By (1)(b), $\mathrm{So}(A\otimes_B Q)\cong A_0S\otimes_B Q$, since Q is M-isotypically semisimple as module over B. Hence, the image of the socle lies in $(A_0S\otimes_S Q)\,\square_{\overline{H}}\,H$.

But $A_0S\otimes_S Q$ is B-semisimple as an epimorphic image of $A_0S\otimes_B Q$. Hence, $A_0S\otimes_S Q\subset\mathrm{So}(A\otimes_S Q)$ and $(A_0S\otimes_B Q)\,\square_{\overline{H}}\,H=\mathrm{So}(A\otimes_S Q)\,\square_{\overline{H}}\,H$, since $(A_0S\otimes_B Q)\,\square_{\overline{H}}\,H$ is semisimple as submodule of $A_0S\otimes_B Q\otimes H$. Therefore, $A_0S\otimes_B Q=\mathrm{So}(A\otimes_S Q)$, since H is faithfully coflat as left $\overline{H}$-comodule.

(E) Proof of (2)(a): Replace So by $A_g\otimes_B M$-So in the proof of part (D).

5.5. Corollary. *In the situation of* 5.4, *assume that M is finitely presented or H is finite dimensional. Let $E:=\mathrm{End}_A(A\otimes_B M)^{\mathrm{op}}$ and $E':=\mathrm{End}_B(M)^{\mathrm{op}}$. Then the extension $E'\subset E$ is an H_{st}-crossed product.*

Proof. By 5.4(1)(a), $S\otimes_B M$ is isomorphic to the M-socle of $A\otimes_B M$. Hence,

$$E\cong\mathrm{Hom}_B(M,A\otimes_B M)\cong\mathrm{Hom}_B(M,S\otimes_B M)\cong\mathrm{End}_S(S\otimes_B M).$$

By [22], 3.11(2), $B\subset S$ is an H_{st}-Galois extension, and S_B is faithfully flat. Hence, 5.5 follows from 3.6.

5.6. COROLLARY. *Assume the situation of* 5.4. *Then*

(1) *If* $X \subset A \otimes_S Q$ *is a non-zero* A*-submodule, then* $X \cap 1 \otimes Q \neq 0$.

(2) $P \mapsto A \otimes_S P$ *defines a bijection between the set of* S*-submodules of* Q *and the set of* A*-submodules of* $A \otimes_S Q$.

(3) $\mathrm{End}_S(Q) \to \mathrm{End}_A(A \otimes_S Q)$, $f \mapsto 1 \otimes f$, *is a ring isomorphism.*

(4) *If* Q *is simple over* S, *then* $A \otimes_S Q$ *is simple over* A.

PROOF. (1) By 1.5(2), the quotients of the B-module filtration of $A \otimes_B Q$ in 2.1 are semisimple. Since $A \otimes_S Q$ is a quotient module of $A \otimes_B Q$, also $A \otimes_S Q$ has a filtration with B-semisimple quotients. Hence, any non-zero B-submodule of $A \otimes_S Q$ has non-zero socle. In particular, there is a simple B-submodule U of $A \otimes_S Q$ such that $U \cong A_g \otimes_B M$ for some group-like element g of H. Then $A_{g^{-1}} U \cong M$ is contained in the M-socle of $A \otimes_S Q$. Hence, by 5.4(2)(a), $X \cap S \otimes_S Q \neq 0$.

(2) By 1.7(3), A_S is faithfully flat. Hence, the mapping $P \mapsto A \otimes_S P$ from S-submodules of Q to A-submodules of $A \otimes_S Q$ is well-defined and injective. To prove the surjectivity, let X be any A-submodule of $A \otimes_S Q$. Define $P := X \cap 1 \otimes_S Q$. Then $A \otimes_S P$ is a submodule of X, and $X/A \otimes_S P$ is isomorphic to a submodule of $A \otimes_S P/Q$ whose intersection with $S \otimes_S P/Q$ is zero. Hence, by part (1), $X = A \otimes_S P$.

(3) follows from 5.4(2), and (4) is a special case of (2).

5.7. COROLLARY. *In the situation of* 5.4 *let*

$$0 = X_0 \subset X_1 \subset \cdots \subset X_n = S \otimes_B M$$

be a Jordan–Hölder composition series of $S \otimes_B M$ *as left* S*-module. Then*

$$0 = A \otimes_S X_0 \subset A \otimes_S X_1 \subset \cdots \subset A \otimes_S X_n \cong A \otimes_B M$$

is a Jordan–Hölder composition series of $A \otimes_B M$ *as left* A*-module.*

PROOF. Since, by 1.7(3), A_S is flat, $A \otimes_S X_i \to A \otimes_S X_{i+1}$ is injective. It remains to show that $A \otimes_S X_{i+1}/X_i$ is A-simple for all i. By definition of the stabilizer, $S \otimes_B M \cong M \otimes H_{st}$ as left modules. In particular, $S \otimes_B M$ is M-isotypically semisimple as module over B. Hence, all quotients X_{i+1}/X_i are M-isotypically semisimple over B. Therefore, by the irreducibility criterion 5.6(4), $A \otimes_S X_{i+1}/X_i$ is A-simple for all i.

6. Applications

A ring R is called *(semi-) primary* if the Jacobson radical $\mathrm{Ra}(R)$ is nilpotent and if $R/\mathrm{Ra}(R)$ is (semi-) simple artinian.

6.1. PROPOSITION. *Assume H is finite dimensional and all simple H-modules are one-dimensional* (*equivalently, the dual coalgebra H^* is pointed*). *Let $B \subset A$ be any* (*not necessarily faithfully flat*) *right H-Galois extension. If B is primary and N is a simple left A-module, then any simple left A-module is A-isomorphic to ${}_\alpha N$ for some algebra homomorphism $\alpha : H \to k$.* (*See* 3.4 *for the definition of ${}_\alpha N$.*)

PROOF. Since H is finite dimensional, A is finitely generated (and projective) over B (cf. [15] (1.8)). Let X be a simple left A-module. Then ${}_B X$ is finitely generated and there exists a B-submodule $U \subset X$ such that X/U is simple. Since B is (semi-) primary, there is a simple B-submodule V of N. But $X/U \cong V$, since B is primary. Hence, $\mathrm{Hom}_B(X, N) \neq 0$. By 3.5, $\mathrm{Hom}_B(X, N)$ is a right H-module. Thus there exists a simple H-submodule W of $\mathrm{Hom}_B(X, N)$. By assumption, W is one-dimensional. Let $W = k\varphi$, $0 \neq \varphi \in W$. Then $\varphi \circ h = \varphi\alpha(h)$ for all $h \in H$, where $\alpha : H \to k$ is the algebra homomorphism defining the one-dimensional representation W. By 3.5(3), $\varphi : X \to {}_\alpha N$ is A-linear. But X, N and ${}_\alpha N$ are simple A-modules. Hence $X \cong {}_\alpha N$.

If R is a finite dimensional algebra, $s(R)$ will denote the number of isomorphism classes of simple right R-modules.

6.2. COROLLARY. *Assume H is finite dimensional and all simple H-modules are one-dimensional. Let M be an indecomposable left B-module of finite length and*

$$A \otimes_B M \cong \bigoplus_{i=1}^{s} V_i^{n_i},$$

$n_i \geqq 1$, V_i indecomposable left A-modules, $V_i \not\cong V_j$ for all $i \neq j$, the A-Krull–Remak–Schmidt decomposition of the induced module. If M is stable, then

$$s \leqq s(H).$$

In particular, if H is a local algebra, then $A \otimes_B M$ is A-isomorphic to V^n, for some indecomposable left A-module V and some natural number $n \geqq 1$.

PROOF. Since M is stable, $A \otimes_B M \cong M \otimes H$ as left B-modules. In particular, $A \otimes_B M$ is an A-module of finite length. Let $E' \subset E$ be the dual endomorphism rings as in Section 3. Then $E' \subset E$ is an H-Galois extension (even an H-crossed product) by 3.6. Now endomorphism rings of modules of finite length are semiprimary (cf. [20], 2.9.10). E' is local, hence primary, since M is indecomposable. By 6.1, the number of isomorphism classes of simple left E-modules is $\leqq s(H)$. Therefore, in a Krull–Remak–Schmidt decomposition of E, the number of isomorphism classes of the indecomposable direct summands is $\leqq s(H)$. This proves 6.2, since the functor $X \mapsto \mathrm{Hom}_A(N, X)$, $N := A \otimes_B M$, from the category of left A-modules which are isomorphic to A-direct summands of N^t for some t to the category of finitely generated and projective left E-modules is an equivalence (cf. [4], (6.3)).

6.3. REMARK. (1) 6.2 contains as a very special case the following variant of Green's theorem ([4], (19.22)) for stable modules in [14]:

Let G be a finite group, G' a normal subgroup and $\bar{G}$ the quotient group G/G'. Consider the special case of 6.2 where $B := k[G'] \subset A := k[G]$ and $H := k[\bar{G}]$ are the modular group algebras, $\mathrm{char}(k) = p \neq 0$. If $\bar{G}$ is a p-group and M is an indecomposable stable G'-module, then, by 6.2, the induced module is isomorphic to V^n for some indecomposable G-module and some $n \geqq 1$. If M is absolutely indecomposable, then $n = 1$ by Green's theorem. But in general, it is well-known that the case $n \neq 1$ occurs.

(2) In the case of finite group schemes in example (3) of the introduction, 6.2 was proved by Voigt [28], 12.8, for H local and k algebraically closed. Contrary to the constant case of Green's theorem, $n \neq 1$ is possible even over an algebraically closed field (see [28], 12.9).

(3) In the situation of the previous remark, one cannot omit the assumption that M is stable. An example, where different isomorphism types occur in the Krull–Remak–Schmidt decomposition of the induced module of an unstable module (although H is local and k is algebraically closed), was given by M. Josek. Examples of this type and another proof of 6.1 which works for more general Hopf algebras will appear in his thesis at the university of Munich.

6.4. COROLLARY. *Assume H is finite dimensional, cocommutative and pointed, and all simple H-modules are one-dimensional. Let M be a simple left B-module such that k is the center of* $\mathrm{End}_B(M)$.

(1) *The number of isomorphism classes of the indecomposable direct summands of an A-Krull–Remak–Schmidt decomposition of $A \otimes_B M$ is $\leqq s(H)$.*

(2) *All A-Jordan–Hölder composition factors of* $A \otimes_B M$ *are isomorphic as B-modules. If H is local, then they are A-isomorphic.*

PROOF. Define E' and E as before. Let H_{st} be the stabilizer of M and $S := A(H_{st})$. By 5.5, $E' \subset E$ is an H_{st}-crossed product. Since H_{st} is a Hopf subalgebra of H, all simple H_{st}-modules are one-dimensional (H_{st}^* is pointed as quotient coalgebra of the pointed Hopf algebra H^*). Hence, (1) is proved in the same way as 6.2.

To prove (2), let V_1 and V_2 be A-composition factors of $A \otimes_B M$. By 5.7, there are S-composition factors U_1 and U_2 of $S \otimes_B M$ such that $V_i \cong A \otimes_S U_i$ over A for $i = 1, 2$. Since H_{st} is the stabilizer of M, $S \otimes_B M \cong M \otimes H_{st}$ over B. Hence, $S \otimes_B M$ is a "progenerator" in the category $\mathscr{C}$ of all left S-modules which are M-isotypically semisimple as modules over B. Therefore,

$$X \mapsto \mathrm{Hom}_S(S \otimes_B M, X)$$

is an equivalence from $\mathscr{C}$ to the category of all left E_S-modules, where $E_S := \mathrm{End}_S(S \otimes_B M)^{\mathrm{op}}$ (cf. [1], (1.3), [28], p. 121). By 6.1, all simple left E_S-modules have the same dimension. If H is local, then H_{st} is local, and all simple left E_S-modules are isomorphic.

Therefore, any two S-composition factors of $S \otimes_B M$ are isomorphic over B. If H is local, they are isomorphic over S. Thus, $U_1 \cong U_2$ over B. Now it follows from 1.8(2) that $A \otimes_B U_i \cong (A \otimes_S U_i)^n$, $i = 1, 2$, as B-modules, $n := [H_{st} : k]$. Then, by Krull–Remak–Schmidt, $A \otimes_S U_1$ and $A \otimes_S U_2$ are isomorphic over B. Hence, V_1 and V_2 are B-isomorphic. If H is local, $U_1 \cong U_2$ over S and V_1 and V_2 are A-isomorphic.

6.5. COROLLARY. *Assume H is finite dimensional, cocommutative and pointed, and all simple H-modules are one-dimensional. Assume A is finite dimensional and B is semisimple with splitting field k. Then all simple A-modules belonging to the same block of A are isomorphic over B. If H is local, then all blocks of A are primary.*

PROOF. Let $B = \bigoplus_i Be_i$, $e_i^2 = e_i \in B$, Be_i simple over B for all i. Decompose $Ae_i = \bigoplus_j Ae_{ij}$, where the Ae_{ij} are indecomposable left A-modules. It suffices to show that all A-composition factors of any one Ae_{ij} are isomorphic over B resp. over A in case H is local. This follows from 6.4(2), applied to $M = Be_i$.

6.6. COROLLARY. *Assume A and H are finite dimensional and H is irreducible (as coalgebra, i.e. the dual algebra is local). Let* e_i, $1 \leqq i \leqq n$, *be central* *idempotents of B such that $1 = \Sigma_i e_i$ and $B = \bigoplus_i Be_i$ is the block decomposition of B. Then*:

(1) *For all i, e_i is central in A and $Be_i \subset Ae_i$ is a right H-Galois extension.*

(2) *If H is local, then $A = \bigoplus_i Ae_i$ is the block decomposition of A, and if Be_i is primary for some i, then Ae_i is primary.*

(3) *If all simple H-modules are one-dimensional, and if all blocks of B are primary, then all simple A-modules belonging to the same block of A are isomorphic over B.*

PROOF. (1) If $e^2 = e \in B$ is central in A, then $Ae \to Ae \otimes H$, $ae \mapsto \Sigma\, a_0 e \otimes a_1$, is an H-comodule algebra, and $Be \subset Ae$ is a right H-Galois extension. Hence it remains to prove that $Ae_iAe_j = 0$ for $i \neq j$.

By the general assumption in Section 1, B is a left and right B-direct summand in A (cf. [15] (1.9)). Hence, if e is any idempotent in B, then Be is a B-direct summand of A, which is stable by 3.7 and 3.3, since H is irreducible.

Thus, $Ae \cong A \otimes_B Be \cong Be \otimes H$ as left B-modules, and Ae and Be have the same B-composition factors.

Now assume $Ae_iAe_j \neq 0$ and $i \neq j$. Then Ae_i and Ae_j have an A- and hence also a B-composition factor in common. Therefore Be_i and Be_j have a common composition factor. But this is a contradiction, since Be_i and Be_j are different block ideals.

(2) Assume Be_i is primary. Since, by (1), e_i is central in A, Ae_i is primary by 6.1. By (1), it remains to prove the following: Let $e^2 = e \in B$ be central in A such that Be is a block in B. Then Ae is a block in A.

Write $Be = \bigoplus_j Bf_j$, where the f_j are orthogonal primitive idempotents in B and $e = \Sigma_j f_j$. Then $Ae = \bigoplus_j Af_j$. For all j, Bf_j is stable and indecomposable. Hence, by 6.2, there is an indecomposable A-module P_j such that $Af_j \cong P_j^{n_j}$ for some n_j. Write $Af_j = \bigoplus_\mu Af_{j\mu}$, where the $f_{j\mu}$ are primitive idempotents in A. Then $P_j \cong Af_{j\mu}$ for all μ. It remains to show that $f_{j\mu}$ and $f_{k\nu}$ lie in the same A-block for all j, k and μ, ν.

Since f_j, f_k lie in the same B-block, one can assume that Af_j, Af_k have an A-composition factor in common, since A_B is flat. Hence, $Af_{j\mu} \cong P_j$ and $Af_{k\nu} \cong P_k$ have a common composition factor.

(3) By (1) and 6.1, all simple left Ae_i-modules are B-isomorphic for all i. This proves the claim, since the blocks of A are the blocks of the Ae_i.

6.7. REMARK. (1) For finite group schemes, see example (3) of the introduction, the above results have the following corollaries (cf. [28], 2.37, 2.41):

(a) If k is algebraically closed, G' linearly reductive and G/G' trigonalizable resp. unipotent, then all blocks of $H(G)$ are full matrix rings over basic resp. local algebras.

(b) If G is infinitesimal and nilpotent, then all blocks of $H(G)$ are full matrix rings over local rings.

PROOF. (a) follows from 6.5. Note that a primary algebra is isomorphic to a full matrix algebra over a local ring (cf. [4], 6.11), Similarly, if k is a splitting field of the finite dimensional k-algebra R and if all simple R-modules belonging to the same block have the same k-dimension, then R is isomorphic to a full matrix ring over a basic algebra S (i.e. $S/\mathrm{Ra}(S) \cong k^m$ for some m).

(b) By the structure theorem in [6], IV, §4, 1.11, there is a (multiplicative) central subgroup G' of G such that G/G' is unipotent. Hence, $H(G/G')$ is irreducible and local, and all blocks of $H(G')$ are primary. Therefore, (b) follows from 6.6.

(2) However, the block structure of the cocommutative Hopf algebras $H = H(G)$ in (1) can be derived more easily by looking at the principal block containing the trivial H-module k (defined by ε). As in [28], 2.37, cf. [13], [17], it is sufficient to show for any semisimple normal Hopf subalgebra H' of H that the principal blocks of H and of $\bar{H} := H/HH'^+$ are isomorphic. (Then the kernel of the principal block is the largest linearly reductive normal subgroup of G.) This fact can be proved directly in the following way (over arbitrary fields): Since H' is semisimple, by [16], Prop. 3 and 4, there is a left and right integral e of H' (i.e. $xe = \varepsilon(x)e = ex$ for all $x \in H'$) such that $\varepsilon(e) = 1$. Then $e^2 = e$, and e is central in H: For all $x \in H$, $xe - ex \in H'^+H = HH'^+$, since $\bar{e} = \bar{1}$ in $\bar{H}$; hence, $xe - ex$ is annihilated by e from the left and the right side, thus $ex = exe = xe$. Now it is clear that $eH \to \bar{H}$, $ex \mapsto \bar{x}$, is an algebra isomorphism, since any $x \in H'^+H$ is annihilated by e. But the principal block is contained in eH, since there is an element $h \in H$ such that eh is a non-zero left integral of H (cf. [19], 2.7). In particular, if $\bar{H}$ is local, then $eH \cong \bar{H}$ is the principal block.

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